Exercises Lecture 16 - Solutions Part I

Ex III.4.1 Since $E(X^2) = E(|X|^2) < \infty$ then Prop. III.3.4 implies $E(|X|) < \infty$. But $E(|X|) = E(X_+) + E(X_-)$ and so $E(X_+) < \infty$, $E(X_-) < \infty$ which implies $E(X) = E(X_+) - E(X_-) < \infty$

Put $Z = XY$ and note

$$|XY| = |X|\,|Y| \le (\max\{|X|, |Y|\})^2 \le X^2 + Y^2$$

so $E(|Z|) \le E(X^2 + Y^2) = E(X^2) + E(Y^2) < \infty$ and so $E(Z) = E(XY)$ is finite.

Therefore $Var(X_i) = E(X_i^2) - (E(X_i))^2 < \infty$ and $Cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$ is finite which implies that all entries of Σ_X are finite and so $\Sigma_{\underset{\sim}{X}} \in \mathbb{R}^{k \times k}$.

Ex III.4.2 $Cov(X, Y) = E((X - E(X))(Y - E(Y)))$

$$= E(XY - XE(Y) - E(X)Y + E(X)E(Y))$$

$$= E(XY) - E(X)E(Y) - \cancel{E(X)E(Y)} + \cancel{E(X)E(Y)}$$

$$\Sigma_{\underset{\sim}{X}} = E((\underset{\sim}{X} - \underset{\sim}{\mu}_X)(\underset{\sim}{X} - \underset{\sim}{\mu}_X)')$$

$$= E(\underset{\sim}{X}\underset{\sim}{X}' - \underset{\sim}{X}\underset{\sim}{\mu}_X' - \underset{\sim}{\mu}_X\underset{\sim}{X}' + \underset{\sim}{\mu}_X\underset{\sim}{\mu}_X')$$

$= E(\underset{\sim}{X}\underset{\sim}{X}') - E(\underset{\sim}{X})\underset{\sim}{\mu}_X' - \underset{\sim}{\mu}_X E(\underset{\sim}{X}') + \underset{\sim}{\mu}_X \underset{\sim}{\mu}_X'$

$= E(\underset{\sim}{X}\underset{\sim}{X}') - \underset{\sim}{\mu}_X \underset{\sim}{\mu}_X' \cancel{- \underset{\sim}{\mu}_X \underset{\sim}{\mu}_X' + \underset{\sim}{\mu}_X \underset{\sim}{\mu}_X'}$

Ex. III.4.3 If $A \in \mathbb{R}^{k \times l}$, $X \in \mathbb{R}^{k \times l}$ then $A + X\ (a_{ij} + X_{ij})$ and so $E(A+X) = (a_{ij} + E(X_{ij}))$ $= A + E(X)$.

If $B \in \mathbb{R}^{p \times k}$ then $BX = \left(\sum_{m=1}^{k} b_{im} X_{mj}\right)$ so $E(BX) = \left(E\left(\sum_{m=1}^{k} b_{im} X_{mj}\right)\right) = \left(\sum_{m=1}^{k} b_{im} E(X_{mj})\right) = BE(X)$.
If $C \in \mathbb{R}^{l \times q}$ then $XC = \left(\sum_{m=1}^{l} c_{mj} X_{im}\right)$ and so $E(XC) = E(X)C$.

Putting all of these together implies $E(A + BXC) = A + BE(X)C$.

Ex III.4.4 $E(\underset{\sim}{X}'\underset{\sim}{X}) = E\left(\sum_{i=1}^{k} X_i^2\right) = \sum_{i=1}^{k} E(X_i^2)$

$= \sum_{i=1}^{k} \left(Var(X_i) + \mu_{X_i}^2\right) = \sum_{i=1}^{k} (\sigma_{ii} + \mu_i^2)$

Ex III.4.5 Recall $X_i \sim$ binomial (n, p_i) and so

$E(X_i) = np_i$, $Var(X_i) = np_i(1-p_i)$. Also

$cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$

$= E(X_i X_j) - n^2 p_i p_j$

and $(X_i, X_j) \sim$ multinomial $(n, p_i, p_j, 1 - p_i - p_j)$ so

$$E(X_i X_j) = \sum_{\substack{x_i, x_j \\ 0 \le x_i + x_j \le n}} x_i x_j \binom{n}{x_i \; x_j \; n-x_i-x_j} p_i^{x_i} p_j^{x_j} (1-p_i-p_j)^{n-x_i-x_j}$$

$$= \sum_{x_i=0}^{n} \sum_{x_j=0}^{n-x_i} x_i x_j \frac{n!}{x_i!\, x_j!\, (n-x_i-x_j)!} p_i^{x_i} p_j^{x_j} (1-p_i-p_j)^{n-x_i-x_j}$$

$$= n(n-1) p_i p_j \sum_{x_i=1}^{n-1} \sum_{x_j=1}^{n-1-x_i} \frac{(n-2)!}{(x_i-1)!\,(x_j-1)!\,(n-2-(x_i-1)-(x_j-1))!}$$
$$\times\, p_i^{x_i-1} p_j^{x_j-1} (1-p_i-p_j)^{n-2-(x_i-1)-(x_j-1)}$$

and putting $z_i = x_i - 1$, $z_j = x_j - 1$ in the sum

$$= n(n-1) p_i p_j \sum_{z_i=0}^{n-2} \sum_{z_j=0}^{n-2-z_i} \binom{n-2}{z_i \; z_j \; n-2-z_i-z_j} \times p_i^{z_i} p_j^{z_j} (1-p_i-p_j)^{n-2-z_i-z_j}$$

$\shortparallel$

sum of all multinomial $(n-2, p_i, p_j, 1-p_i-p_j)$ probabilities

$\shortparallel$ 1

$= n(n-1) p_i p_j$ and so $Cov(X_i, X_j) = n(n-1)p_i p_j - n^2 p_i p_j = -n p_i p_j$

Therefore $\underset{\sim}{\mu_X} = \begin{pmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{pmatrix}$, $\underset{\sim}{\Sigma_X} = \begin{pmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_k \\ -np_1p_2 & np_2(1-p_2) & & \\ \vdots & & \ddots & \\ -np_1p_k & & & np_k(1-p_k) \end{pmatrix}$

III.4.6

(i) We need $E(X^2) < \infty$, $E(Y^2) < \infty$ so that $\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$ exists and is finite. Also we need $\mathrm{Var}(X) > 0$, $\mathrm{Var}(Y) > 0$.

(ii) $\mathrm{cov}(a+bX, c+dY)$

$$= E((a+bX)(c+dY)) - E(a+bX)E(c+dY)$$

$$= E(ac+adY+bcX+bdXY) - (a+bE(X))(c+dE(Y))$$

$$= \cancel{ac} + \cancel{adE(Y)} + \cancel{bcE(X)} + bdE(XY) - \cancel{ac} - \cancel{adE(Y)} - \cancel{bcE(X)} - bdE(X)E(Y)$$

$$= bd\,(E(XY) - E(X)E(Y)) = bd\,\mathrm{cov}(X,Y)$$

$$\mathrm{Var}(a+bX) = E((a+bX-a-bE(X))^2)$$

$$= b^2 E((X-E(X))^2) = b^2\,\mathrm{Var}(X)$$ and similarly

$$\mathrm{Var}(c+dY) = c^2\,\mathrm{Var}(Y)$$

Therefore $$\mathrm{CORR}(a+bX, c+dY) = \frac{\cancel{bd}\,\mathrm{cov}(X,Y)}{\cancel{b}SD(X)\,\cancel{d}SD(Y)}$$

$$= \mathrm{CORR}(X,Y).$$

When $b=0$ $\mathrm{Var}(a+bX)=0$ and so $\mathrm{CORR}(X,Y)$ is not defined.

When $b<0, d>0$ $$\mathrm{CORR}(a+bX, c+dY) = \frac{bd\,\mathrm{cov}(X,Y)}{|b|SD(X)\,SD(Y)}$$

$$= -\mathrm{CORR}(X,Y).$$

when $b<0, d<0$ $\text{CORR}(a+bX, c+dY)$

$$= \frac{bd\,\text{COV}(X,Y)}{|b|\,SD(X)\,|d|\,SD(Y)} = \frac{\text{COV}(X,Y)}{SD(X)\,SD(Y)} = \text{CORR}(X,Y).$$

(iii) $\text{COV}(X,Y) = \text{COV}(X, a+bX)$

$= E\left((X-E(X))(a+bX-a-bE(X))\right)$

$= E\left((X-E(X))\,b(X-E(X))\right) = b\,\text{Var}(X)$ and

$\text{Var}(Y) = b^2\,\text{Var}(X)$ so

$$\text{CORR}(X,Y) = \frac{b\,\text{Var}(X)}{|b|\,\text{Var}(X)} = \text{sgn}\,b = \begin{cases} 1 & b>0 \\ -1 & b<0 \end{cases}.$$

(iv) $E(X) = \frac{1}{2}$, $E(X^2) = \int_0^1 x^2\,dx = \frac{1}{3}$

$E(XY) = E(XX^2) = E(X^3) = \frac{1}{4}$, $E(Y^2) = E(X^4) = \frac{1}{5}$

$$\text{CORR}(X,Y) = \frac{\frac{1}{4} - \frac{1}{2}\frac{1}{3}}{\sqrt{\frac{1}{3}-\frac{1}{2^2}}\sqrt{\frac{1}{5}-\frac{1}{9}}} = \frac{1/12}{\sqrt{1/12}\sqrt{4/45}} = \sqrt{\frac{45}{48}}$$

which is not equal to 1.

(v) $E(X) = 0$, $E(X^2) = \frac{1}{2}\int_{-1}^1 x^2\,dx = \frac{1}{2}\frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3}$

$E(Y) = E(X^2) = \frac{1}{3}$, $E(Y^2) = \frac{1}{2}\int_{-1}^1 x^4\,dx = \frac{1}{2}\frac{1}{5}x^5\Big|_{-1}^{1}$

$= \frac{1}{5}$

$E(XY) = E(X^3) = 0$. Therefore

$\text{COV}(X,Y) = 0 - 0\cdot\frac{1}{3} = 0$ so $\text{CORR}(X,Y) = 0$

but $P(X \le x, Y \le y) = P(X \le x, X^2 \le y)$

$\overset{x=y=\frac{1}{2}}{=} \quad P(X \le \tfrac{1}{2}, X^2 \le \tfrac{1}{2}) = P(-\tfrac{1}{\sqrt{2}} \le X \le \tfrac{1}{2})$

$= \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{\sqrt{2}})$ while $P(X \le \tfrac{1}{2}) = \tfrac{1}{2}\tfrac{3}{2} = \tfrac{3}{4}$

$P(X^2 \le \tfrac{1}{2}) = P(-\tfrac{1}{\sqrt{2}} \le X \le \tfrac{1}{\sqrt{2}}) = \tfrac{1}{\sqrt{2}}$ so

$P(X \le \tfrac{1}{2}, X^2 \le \tfrac{1}{2}) \neq P(X \le \tfrac{1}{2})\, P(X^2 \le \tfrac{1}{2})$

and so X and Y are not stat. ind. but they are uncorrelated.