Solutions to Exercises - Lecture 13

Ex III.3.1 Since $X \sim N(\mu,\sigma^2)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Then $Z^3 = \frac{1}{\sigma^3}(X-\mu)^3 = \frac{1}{\sigma^3}\left(X^3 - \binom{3}{1}\mu X^2 + \binom{3}{2}\mu^2 X - \mu^3\right)$

$$= \frac{1}{\sigma^3}\left(X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\right) \quad \text{so}$$

$$E(Z^3) = \frac{1}{\sigma^3}\left(E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3\right)$$

$$= \frac{1}{\sigma^3}\left(E(X^3) - 3\mu(\sigma^2+\mu^2) + 3\mu^3 - \mu^3\right)$$

$$= \frac{1}{\sigma^3}\left(E(X^3) - 3\mu\sigma^2 - 3\mu^3 + 3\mu^3 - \mu^3\right)$$

$$= \frac{1}{\sigma^3}\left(E(X^3) - 3\mu\sigma^2 - \mu^3\right)$$

and $E(Z^3) = \int_{-\infty}^{\infty} z^3 \varphi(z)\,dz = 0$ since $(-z)^3\varphi(-z)$

$= -z^3\varphi(z)$. Therefore $E(X^3) = \mu^3 + 3\mu\sigma^2$

Also $Z^4 = \frac{1}{\sigma^4}\left(X^4 - \binom{4}{1}\mu X^3 + \binom{4}{2}\mu^2 X^2 - \binom{4}{3}\mu^3 X + \mu^4\right)$

$$= \frac{1}{\sigma^4}\left(X^4 - 4\mu X^3 + 6\mu^2 X^2 - 4\mu^3 X + \mu^4\right) \quad \text{so}$$

$$E(Z^4) = \frac{1}{\sigma^4}\left(E(X^4) - 4\mu(\mu^3 + 3\mu\sigma^2) + 6\mu^2(\sigma^2+\mu^2) - 4\mu^4 + \mu^4\right)$$

$$= \frac{1}{\sigma^4}\left(E(X^4) - 6\mu^2\sigma^2 - \mu^4\right)$$

$E(Z^4) = \int_{-\infty}^{\infty} z^4 \varphi(z)\,dz$ putting $u = z^3$, $dv = z\varphi(z)$, $du = 3z^2$, $v = -\varphi(z)$

$= -z^3\varphi(z)\Big|_{-\infty}^{\infty} + 3\int_{-\infty}^{\infty} z^2 \varphi(z) = 3$ and so

$E(X^4) = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$

Ex III.3.2 $E(X_+) = \int_0^\infty x \frac{1}{\pi(1+x^2)}\,dx$

$= \frac{1}{2\pi} \ln(1+x^2) \Big|_0^\infty = \infty - 0 = \infty$ and similarly

$E(X_-) = \int_{-\infty}^0 (-x) \frac{1}{\pi(1+x^2)}\,dx = \infty$. Therefore

$E(X)$ is not defined.

Ex III.3.3 E&R 3.1.17 Let X = # of tails until first H

(a) $E(Y) = \sum_{y=0}^{100} y(1-\theta)^y \theta + 100\left(\sum_{y=101}^{\infty} (1-\theta)^y \theta\right)$

$= -\theta(1-\theta) \sum_{y=0}^{100} \frac{d(1-\theta)^y}{d\theta} + 100\,\theta(1-\theta)^{101} \sum_{y=0}^{\infty} (1-\theta)^y$

$= -\theta(1-\theta) \frac{d}{d\theta} \sum_{y=0}^{100} (1-\theta)^y + \frac{100\,\theta(1-\theta)^{101}}{1-(1-\theta)}$

$= -\theta(1-\theta) \frac{d}{d\theta} \left(\frac{1-(1-\theta)^{101}}{1-(1-\theta)}\right) + 100(1-\theta)^{101}$

$= -\theta(1-\theta) \frac{d}{d\theta} \left\{\left[1-(1-\theta)^{101}\right]\theta^{-1}\right\} + 100(1-\theta)^{101}$

$= -\theta(1-\theta)\left[101(1-\theta)^{100}\theta^{-1} - \left(1-(1-\theta)^{101}\right)\theta^{-2}\right] + 100(1-\theta)^{101}$

$= \frac{(1-\theta)}{\theta}\left[1 - 101(1-\theta)^{100}\theta - (1-\theta)^{101}\right] + 100(1-\theta)^{101}$

$= \frac{1-\theta}{\theta}\left[1-(1-\theta)^{101}\right] - (1-\theta)^{101}$

(b) $E(Y-X) = E(Y) - E(X) = -\frac{(1-\theta)^{102}}{\theta} - (1-\theta)^{101}$ since $E(X) = \frac{1-\theta}{\theta}$

Ex III, 3.4

E+R 3.1.22 Let X = # of tails until the r-th H so if $x_1, \dots, x_r \overset{iid}{\sim}$ geometric (θ) we have that $X_1 + \dots + X_r \sim$ Negative binomial (r, θ). Therefore $E(X) = E(X_1 + \dots + X_r) = rE(X_1) = r(1-\theta)/\theta$.

E+R 3.3.18 If $X \sim$ geometric (θ) then $E(X) = (1-\theta)/\theta$ and

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1)(1-\theta)^x \theta = \theta(1-\theta)^2 \sum_{x=0}^{\infty} x(x-1)(1-\theta)^{x-2}$$

$$= \theta(1-\theta)^2 \sum_{x=0}^{\infty} \frac{d^2}{d\theta^2}(1-\theta)^x = \theta(1-\theta)^2 \frac{d^2}{d\theta^2} \sum_{x=0}^{\infty} (1-\theta)^x$$

$$= \theta(1-\theta)^2 \frac{d^2}{d\theta^2} \theta^{-1} = \frac{2(1-\theta)^2}{\theta^2}$$

Therefore $Var(X) = E(X(X-1)) - E(X)(E(X)-1)$

$$= \frac{2(1-\theta)^2}{\theta^2} - \frac{1-\theta}{\theta}\left(\frac{1-\theta}{\theta} - 1\right) = \frac{(1-\theta)^2}{\theta^2} + \frac{1-\theta}{\theta} = \frac{1-\theta}{\theta^2}$$

E+R 3.3.19 Use $X_1, \dots, X_r \overset{iid}{\sim}$ geometric (θ) then $X = X_1 + \dots + X_r \sim$ negative binomial (r, θ) so $Var(X) = r\,Var(X_1) = r\frac{1-\theta}{\theta^2}$.

Ex III.3.5

B&R 3.2.16

$$E(X) = \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)} \underbrace{\int_0^\infty \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{\alpha+1-1} e^{-\lambda x} dx}_{=1}$$

$$= \frac{\alpha}{\lambda} \quad \text{since } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

B&R 3.3.20

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \frac{\Gamma(\alpha+2)}{\lambda^2 \Gamma(\alpha)} \underbrace{\int_0^\infty \frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{\alpha+2-1} e^{-\lambda x} dx}_{=1}$$

$$= \frac{(\alpha+1)\alpha}{\lambda^2} \quad \text{. Therefore } Var(X) = \frac{(\alpha+1)\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}.$$

Ex III.3.6 E+R 3.2.22

$$E(X) = \int_0^1 x \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a+1-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)} = \frac{a}{a+b}$$

E+R 3.3.24

$$E(X^2) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} = \frac{a(a+1)}{(a+b+1)(a+b)}$$

$$\therefore \operatorname{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{a(a+1)}{(a+b+1)(a+b)} - \frac{a^2}{(a+b)^2}$$

$$= \frac{a}{a+b}\left[\frac{a+1}{a+b+1} - \frac{a}{a+b}\right]$$

$$= \frac{a}{a+b}\left[\frac{(a+1)(a+b) - a(a+b+1)}{(a+b)(a+b+1)}\right]$$

$$= \frac{ab}{(a+b)^2(a+b+1)}$$