Appendix A

Mathematical Background

To understand this book, it is necessary to know certain mathematical subjects listed below. Because it is assumed the student has already taken a course in calculus, topics such as derivatives, integrals, and infinite series are treated quite briefly here. Multivariable integrals are treated in somewhat more detail.

A.1 Derivatives

From calculus, we know that the *derivative* of a function f is its instantaneous rate of change:

$$f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In particular, the reader should recall from calculus that

$$\tfrac{d}{dx} 5 = 0, \qquad \tfrac{d}{dx} x^3 = 3x^2, \qquad \tfrac{d}{dx} x^n = n x^{n-1},$$

$$\tfrac{d}{dx} e^x = e^x, \qquad \tfrac{d}{dx} \sin x = \cos x, \qquad \tfrac{d}{dx} \cos x = -\sin x,$$

etc. Hence, if $f(x) = x^3$, then $f'(x) = 3x^2$ and, e.g., $f'(7) = 3\,7^2 = 147$.

Derivatives respect addition and scalar multiplication, so if f and g are functions and C is a constant, then

$$\frac{d}{dx}(C\ f(x) + g(x)) = C\ f'(x) + g'(x).$$

Thus,

$$\frac{d}{dx}(5x^3 - 3x^2 + 7x + 12) = 15x^2 - 6x + 7,$$

etc.

Finally, derivatives satisfy a *chain rule*; if a function can be written as a *composition* of two other functions, as in $f(x) = g(h(x))$, then $f'(x) = g'(h(x))\, h'(x)$. Thus,

$$\begin{aligned} \tfrac{d}{dx} e^{5x} &= 5e^{5x}, \\ \tfrac{d}{dx} \sin(x^2) &= 2x \cos(x^2), \\ \tfrac{d}{dx} (x^2 + x^3)^5 &= 5(x^2 + x^3)^4 (2x + 3x^2), \end{aligned}$$

etc.

Higher-order derivatives are defined by

$$f''(x) = \frac{d}{dx} f'(x), \; f'''(x) = \frac{d}{dx} f''(x),$$

etc. In general, the rth-order derivative $f^{(r)}(x)$ can be defined inductively by $f^{(0)}(x) = f(x)$ and

$$f^{(r)}(x) = \frac{d}{dx} f^{(r-1)}(x)$$

for $r \geq 1$. Thus, if $f(x) = x^4$, then $f'(x) = 4x^3$, $f''(x) = f^{(2)}(x) = 12x^2$, $f^{(3)}(x) = 24x$, $f^{(4)}(x) = 24$, etc.

Derivatives are used often in this text.

A.2 Integrals

If f is a function, and $a < b$ are constants, then the *integral* of f over the interval $[a, b]$, written

$$\int_a^b f(x)\, dx,$$

represents adding up the values $f(x)$, multiplied by the widths of small intervals around x. That is, $\int_a^b f(x)\, dx \approx \sum_{i=1}^d f(x_i)(x_i - x_{i-1})$, where $a = x_0 < x_1 < \ldots < x_d = b$, and where $x_i - x_{i-1}$ is small.

More formally, we can set $x_i = a + (i/d)(b - a)$ and let $d \to \infty$, to get a formal definition of integral as

$$\int_a^b f(x)\, dx = \lim_{d \to \infty} \sum_{i=1}^d f\,(a + (i/d)(b - a))\,(1/d).$$

To compute $\int_a^b f(x)\, dx$ in this manner each time would be tedious. Fortunately, the *fundamental theorem of calculus* provides a much easier way to compute integrals. It says that if $F(x)$ is any function with $F'(x) = f(x)$, then $\int_a^b f(x)\, dx = F(b) - F(a)$. Hence,

$$\begin{aligned} \textstyle\int_a^b 3x^2\, dx &= b^3 - a^3, \\ \textstyle\int_a^b x^2\, dx &= \tfrac{1}{3}(b^3 - a^3), \\ \textstyle\int_a^b x^n\, dx &= \tfrac{1}{n+1}(b^{n+1} - a^{n+1}), \end{aligned}$$

and

$$\int_a^b \cos x\,dx = \sin b - \sin a,$$
$$\int_a^b \sin x\,dx = -(\cos b - \cos a),$$
$$\int_a^b e^{5x}\,dx = \tfrac{1}{5}(e^{5b} - e^{5a}).$$

A.3 Infinite Series

If $a_1, a_2, a_3, \ldots$ is an infinite sequence of numbers, we can consider the infinite sum (or *series*)

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$$

Formally, $\sum_{i=1}^{\infty} a_i = \lim_{N\to\infty} \sum_{i=1}^{N} a_i$. This sum may be finite or infinite.

For example, clearly $\sum_{i=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots = \infty$. On the other hand, because

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n},$$

we see that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2^i} = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{2^i} = \lim_{N\to\infty} \frac{2^N - 1}{2^N} = 1.$$

More generally, we compute that

$$\sum_{i=1}^{\infty} a^i = \frac{a}{1-a}$$

whenever $|a| < 1$.

One particularly important kind of infinite series is a *Taylor series*. If f is a function, then its Taylor series is given by

$$f(0) + x f'(0) + \frac{1}{2!}x^2 f''(0) + \frac{1}{3!}x^3 f^{(3)}(0) + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!} x^i f^{(i)}(0).$$

(Here $i! = i(i-1)(i-2)\cdots(2)(1)$ stands for i *factorial*, with $0! = 1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc.) Usually, $f(x)$ will be exactly equal to its Taylor series expansion, thus,

$$\begin{aligned}
\sin x &= x - x^3/3 + x^5/5 - x^7/7 + \cdots, \\
\cos x &= 1 - x^2/2 + x^4/4 - x^5/5 + \cdots, \\
e^x &= 1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots, \\
e^{5x} &= 1 + 5x + (5x)^2/2! + (5x)^3/3! + (5x)^4/4! + \cdots,
\end{aligned}$$

etc. If $f(x)$ is a polynomial (e.g., $f(x) = x^3 - 3x^2 + 2x - 6$), then the Taylor series of $f(x)$ is precisely the same function as $f(x)$ itself.

A.4 Matrix Multiplication

A *matrix* is any $r \times s$ collection of numbers, e.g.,

$$A = \begin{pmatrix} 8 & 6 \\ 5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 6 & 2 \\ -7 & 6 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ 3/5 & 2/5 \\ -0.6 & -17.9 \end{pmatrix},$$

etc.

Matrices can be *multiplied*, as follows. If A is an $r \times s$ matrix, and B is an $s \times u$ matrix, then the product AB is an $r \times u$ matrix whose i, j entry is given by $\sum_{k=1}^{s} A_{ik} B_{kj}$, a sum of products. For example, with A and B as above, if $M = AB$, then

$$\begin{aligned} M &= \begin{pmatrix} 8 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 3 & 6 & 2 \\ -7 & 6 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 8\,(3) + 6\,(-7) & 8\,(6) + 6\,(6) & 8\,(2) + 6\,(0) \\ 5\,(3) + 2\,(-7) & 5\,(6) + 2\,(6) & 5\,(2) + 2\,(0) \end{pmatrix} = \begin{pmatrix} -18 & 84 & 16 \\ 1 & 42 & 10 \end{pmatrix}, \end{aligned}$$

as, for example, the (2, 1) entry of M equals $5\,(3) + 2\,(-7) = 1$.

Matrix multiplication turns out to be surprisingly useful, and it is used in various places in this book.

A.5 Partial Derivatives

Suppose f is a function of *two* variables, as in $f(x, y) = 3x^2y^3$. Then we can take a *partial derivative* of f with respect to x, writing

$$\frac{\partial}{\partial x} f(x, y),$$

by varying x while keeping y fixed. That is,

$$\frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

This can be computed simply by regarding y as a constant value. For the example above,

$$\frac{\partial}{\partial x}(3x^2y^3) = 6xy^3.$$

Similarly, by regarding x as constant and varying y, we see that

$$\frac{\partial}{\partial y}(3x^2y^3) = 9x^2y^2.$$

Other examples include

$$\begin{aligned} \frac{\partial}{\partial x}(18e^{xy} + x^6y^8 - \sin(y^3)) &= 18ye^{xy} + 6x^5y^8, \\ \frac{\partial}{\partial y}(18e^{xy} + x^6y^8 - \sin(y^3)) &= 18xe^{xy} + 8x^6y^7 - 3y^2\sin(y^3), \end{aligned}$$

etc.

If f is a function of three or more variables, then partial derivatives may similarly be taken. Thus,

$$\frac{\partial}{\partial x}(x^2y^4z^6) = 2xy^4z^6, \quad \frac{\partial}{\partial y}(x^2y^4z^6) = 4x^2y^3z^6, \quad \frac{\partial}{\partial z}(x^2y^4z^6) = 6x^2y^4z^5,$$

etc.

A.6 Multivariable Integrals

If f is a function of two or more variables, we can still compute integrals of f. However, instead of taking integrals over an interval $[a, b]$, we must take integrals over higher-dimensional *regions*.

For example, let $f(x, y) = x^2y^3$, and let R be the rectangular region given by $R = \{0 \leq x \leq 1, 5 \leq y \leq 7\} = [0, 1] \times [5, 7]$. What is

$$\int_R\int f(x, y)\,dx\,dy,$$

the integral of f over the region R? In geometrical terms, it is the volume under the graph of f (and this is a surface) over the region R. But how do we compute this?

Well, if y is constant, we know that

$$\int_0^1 f(x, y)\,dx = \int_0^1 x^2y^3\,dx = \frac{1}{3}y^3. \tag{A.6.1}$$

This corresponds to adding up the values of f along one "strip" of the region R, where y is constant. In Figure A.6.1, we show the region on integration $R = [0, 1] \times [5, 7]$. The value of (A.6.1), when $y = 6.2$, is $(6.2)^3/3 = 79.443$; this is the area under the curve $x^2(6.2)^3$ over the line $[0, 1] \times \{6.2\}$.

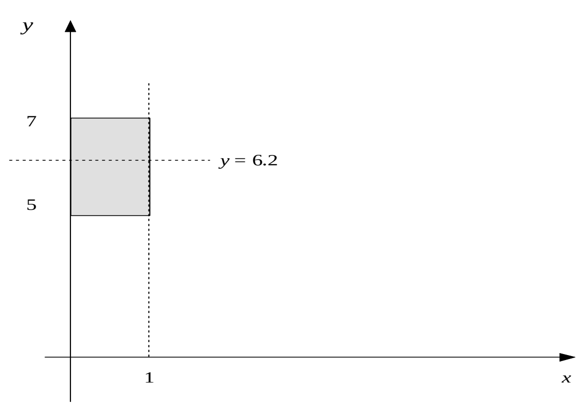

Figure A.6.1: Plot of the region of integration (shaded) $R = [0, 1] \times [5, 7]$ together with the line at $y = 6.2$.

If we then add up the values of the areas over these strips along all different possible y values, then we obtain the overall integral or volume, as follows:

$$\begin{aligned}\int_R\int f(x,y)\,dx\,dy &= \int_5^7\left(\int_0^1 f(x,y)\,dx\right)dy = \int_5^7\left(\int_0^1 x^2y^3\,dx\right)dy \\ &= \int_5^7\left(\frac{1}{3}y^3\right)dy = \frac{1}{3}\,\frac{1}{4}(7^4-5^4) = 148\,.\end{aligned}$$

So the volume under the the graph of f and over the region R is given by 148.

Note that we can also compute this integral by integrating first y and then x, and we get the same answer:

$$\begin{aligned}\int_R\int f(x,y)\,dx\,dy &= \int_0^1\left(\int_5^7 f(x,y)\,dy\right)dx = \int_0^1\left(\int_5^7 x^2y^3\,dy\right)dx \\ &= \int_0^1\left(\frac{1}{4}x^2(7^4-5^4)\right)dx = \frac{1}{3}\,\frac{1}{4}(7^4-5^4) = 148.\end{aligned}$$

Nonrectangular Regions

If the region R is not a rectangle, then the computation is more complicated. The idea is that, for each value of x, we integrate y over only those values for which the point (x, y) is inside R.

For example, suppose that R is the triangle given by $R = \{(x, y) : 0 \le 2y \le x \le 6\}$. In Figure A.6.2, we have plotted this region together with the slices at $x = 3$ and $y = 3/2$. We use the x-slices to determine the limits on y for fixed x when we integrate out y first; we use the y-slices to determine the limits on x for fixed y when we integrate out x first.

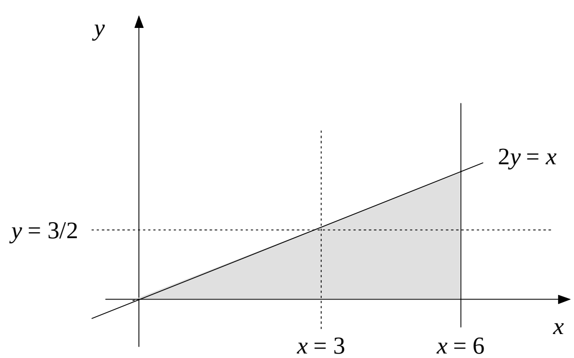

Figure A.6.2: The integration region (shaded) $R = \{(x, y) :\ 0 \le 2y \le x \le 6\}$ together with the slices at $x = 3$ and $y = 3/2$.

Then x can take any value between 0 and 6. However, once we know x, then y can only take values between 0 and $x/2$. Hence, if $f(x, y) = xy + x^6y^8$, then

$$\begin{aligned}
&\int_R\int f(x,y)\,dx\,dy \\
&= \int_0^6 \Big(\int_0^{x/2} f(x,y)\,dy\Big)dx = \int_0^6 \Big(\int_0^{x/2} (xy + x^6y^8)\,dy\Big)\,dx \\
&= \int_0^6 \Big(x\frac{1}{2}((x/2)^2 - 0^2) + (x^6\frac{1}{9}((x/2)^9 - 0^9))\Big)\,dx \\
&= \int_0^6 \Big(\frac{1}{8}x^3 + \frac{1}{4608}x^{15}\Big)\,dx \\
&= \frac{1}{8}\frac{1}{4}(6^4 - 0^4) + \frac{1}{4608}\frac{1}{16}(6^{16} - 0^{16}) \\
&= 3.8264 \times 10^7.
\end{aligned}$$

Once again, we can compute the same integral in the opposite order, by integrating first x and then y. In this case, y can take any value between 0 and 3. Then, for a given value of y, we see that x can take values between 0 and $2y$. Hence,

$$\int_R\int f(x,y)\,dx\,dy = \int_0^3 \Big(\int_0^{2y} f(x,y)\,dx\Big)\,dy = \int_0^3 \Big(\int_0^{2y} (xy + x^6y^8)\,dx\Big)\,dy.$$

We leave it as an exercise for the reader to finish this integral, and see that the same answer as above is obtained.

Functions of three or more variables can also be integrated over regions of the corresponding dimension three or higher. For simplicity, we do not emphasize such higher-order integrals in this book.