Lecture 1 (Review of High School Math: Functions and Models)

Introduction: Numbers and their properties
Addition:

(1) (Associative law) If $a$, $b$, and $c$ are any numbers, then
$$a + (b + c) = (a + b) + c$$

(2) (Existence of an additive identity) If $a$ is any number, then
$$a + 0 = 0 + a = a$$

(3) (Existence of additive inverses) For every number $a$, there is a number $-a$ such that
$$a + (-a) = (-a) + a = 0$$

(4) (Commutative law) If $a$ and $b$ are any numbers, then
$$a + b = b + a$$
Multiplication:

(5) (Associative law) If \(a\), \(b\), and \(c\) are any numbers, then
\[
a \cdot (b \cdot c) = (a \cdot b) \cdot c
\]

(6) (Existence of an multiplicative identity) If \(a\) is any number, then
\[
a \cdot 1 = 1 \cdot a = a
\]

(7) (Existence of multiplicative inverses) For every number \(a \neq 0\), there is a number \(a^{-1}\) such that
\[
a \cdot a^{-1} = a^{-1} \cdot a = 1
\]
(Note: division by 0 is always undefined!)

(8) (Commutative law) If \(a\) and \(b\) are any numbers, then
\[
a \cdot b = b \cdot a
\]

(9) (Distributive law) If \(a\), \(b\), and \(c\) are any numbers, then
\[
a \cdot (b + c) = a \cdot b + a \cdot c
\]
**Definition:** The numbers $a$ satisfying $a > 0$ are called **positive**, while those numbers $a$ satisfying $a < 0$ are called **negative**.

For any number $a$, we define the **absolute value** $|a|$ of $a$ as follows:

$$ |a| = \begin{cases} 
  a, & a \geq 0 \\
  -a, & a \leq 0 
\end{cases} $$

**Note:** $|a|$ is always positive, except when $a = 0$

**Example:**

$$ |\neg2| = 2 $$

$$ f(x) = |x| = \begin{cases} 
  x, & x \geq 0 \\
  -x, & x \leq 0 
\end{cases} $$

$$ |x| \leq 2 \quad \Rightarrow \quad -2 \leq x \leq 2 $$
Theorem (Triangle Inequality): For all numbers $a$ and $b$, we have
\[ |a + b| \leq |a| + |b| \]

Proof:

Note: $a \leq |a|$

\[
\sqrt{|a+b|^2} = (a+b)^2 = a^2 + 2ab + b^2
\]

\[
= |a|^2 + 2a|b| + |b|^2
\]

\[
\leq |a|^2 + 2|a||b| + |b|^2
\]

\[
= \sqrt{(|a| + |b|)^2}
\]

\[
|a + b| \leq |a| + |b|
\]
Exercises

1. Prove the following:
   (a) \( x^2 - y^2 = (x - y)(x + y) \)

   (b) \( (x \pm y)^2 = x^2 \pm 2xy + y^2 \)

   (c) \( x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2) \)
2. What is wrong with the following «proof»?

Let

\[ x = y \]

then

\[ x^2 = xy \]
\[ x^2 - y^2 = xy - y^2 \]
\[ (x + y)(x - y) = y(x - y) \]
\[ x + y = y \]
\[ 2y = y \]
\[ 2 = 1 \]

\( \text{Wrong!} \)

Cannot divide by 0.
What types of numbers are there?...

The simplest numbers are the «counting numbers»:

1, 2, 3, ...

We call them natural numbers and denote by \( \mathbb{N} \).

The most basic property of \( \mathbb{N} \) is the principle of «mathematical induction». 
**Mathematical Induction:** Suppose $P(n)$ means that the property $P$ holds for the number $n$. Then $P(n)$ is true for all natural numbers $n$ provided that

1. $P(1)$ is true
2. Whenever $P(k)$ is true, $P(k + 1)$ is true.

A standard analogy is a string of dominoes which are arranged in such a way that if any given domino is knocked over then it in turn knocks over the next one.

This analogy is a good one but it is only an analogy, and we have to remember that in the domino situation there is only a **finite number** of dominoes.
Example: Show that \( 1 + \ldots + n = \frac{n(n+1)}{2} \) \((*)\)

Solution:

(1) Show \((*)\) is true for \(n=1\)

\[
\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1
\]

(2) Assume \((*)\) is true for \(n=k\), i.e.

\[1 + \ldots + k = \frac{k(k+1)}{2}\]

Need to prove for \(n=k+1\), i.e.

\[
\frac{1 + \ldots + k + (k+1)}{2} = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}
\]

\[
\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}
\]
Exercise

Prove by induction on $n$ that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$ (note that if $r = 1$, you can easily calculate the sum)
Other numbers:

**Integers**: ..., -2, -1, 0, 1, 2, ... This set is denoted by \( \mathbb{Z} \).

**Rational numbers**: \( \frac{m}{n}, \ n \neq 0, \ m, n \in \mathbb{Z} \). This set is denoted by \( \mathbb{Q} \).

**Real numbers**: denoted by \( \mathbb{R} \).

Real numbers include rational and **irrational numbers** (e.g. \( \pi \) or \( \sqrt{2} \), i.e. numbers that can be represented by infinite decimals).

Why is \( \sqrt{2} \) irrational? Assume it is rational.

\[
\sqrt{2} = \frac{a}{b} \rightarrow \text{irreducible}
\]

\[
\sqrt{2}b = a
\]

\[
2b^2 = a^2
\]

So \( a \) is even, i.e.

\[
a = 2k, \ k \in \mathbb{Z}
\]

\[
2b^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b \text{ is even, i.e. } b = 2m
\]

Contradiction to irreducibility of \( \frac{a}{b} \).
Set notation and set operations

Definition: A set \( A \) is a collection of objects which are called elements or members.

Example: \( A = \{-1, 0, 1, 2\} \)

Symbols that we shall use:

\( x \in A \) (\( x \) belongs to \( A \))

\[-1 \in A\]

\( x \notin A \) (\( x \) does not belong to \( A \))

\[5 \notin A\]
Subset: \( A \subset B \)

\( \forall x \in A \Rightarrow x \in B \)

Venn Diagram:

Complement: \( A^c \)

\( x \in A^c \Rightarrow x \notin A \)

\( A = \{-1, 0, 1, 2\} \)

\( A^c = \mathbb{R} \setminus \{-1, 0, 1, 2\} \)
Union: $A \cup B = \{ x : x \in A \text{ or } x \in B \}$

$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

Intersection: $A \cap B = \{ x : x \in A \text{ and } x \in B \}$

$A \cap B = \{3, 4\}$

Empty set: $\emptyset$

$A \cap B = \emptyset$

$A \cup B$ disjoint
Intervals: \([a, b], \ (a, b), \ [a, b), \ (a, b]\)

- closed interval \([a, b]\)
- open interval \((a, b)\)
- half-closed interval \([a, b)\)
- half-closed interval \((a, b]\)

\[\exists x \quad x \in [-1, 3]\]

\[x \in (-1, 2)\]
Example:

\((-1, 4) \cap (0, 12) = (0, 4)\)

\((-\infty, -3) \cup (-4, \infty) = \mathbb{R} = (-\infty, \infty)\)

\([0, 7]^c = (-\infty, 0] \cup (7, \infty)\)
Solving inequalities

Example: Solve $2 - 3x > 8$.

\[
2 - 8 > 3x \\
3x < -6 \\
x < -2
\]

Express the answer as an interval and graphically.

\((-\infty, -2)\)
Example: \(x^2 - 3x + 3 \geq 1\)

\[
x^2 - 3x + 2 \geq 0
\]

\[(x-2)(x-1) \geq 0\]

\[\begin{align*}
  &+ &+ &+ \\
  &1 &2 \\
  x-2 &< 0 & x-2 &> 0 \\
  x-1 &< 0 & x-1 &> 0
\end{align*}\]

\[x \in (-\infty, 1] \cup [2, \infty)\]
Example: Solve $|x - 3| \leq 2$

$$-2 \leq x - 3 \leq 2$$

$$1 \leq x \leq 5$$

$$x \in [1, 5]$$
Functions

What is a function?
- A **function** is a rule which assigns, to each of certain real numbers, some other real number.

**Notation:** \( f(x) \).
Example: The rule which assigns to each number the cube of that number:

\[ f(x) = x^3 \]
Using notations:

- A function $f$ is a rule that assigns to each element $x$ from some set $D$ exactly one element, $f(x)$, in a set $E$.
- $D$ is a set of real numbers, called the **domain** of the function.
- $E$ is a set of real numbers, called the **range** of the function, it is the set of all possible values of $f(x)$ defined for every $x$ in the domain.
- We call $x$ an **independent** variable, and $y = f(x)$ a **dependent** variable.

Examples: Find domain and range in interval notation.

1. $f(x) = x^2$
   
   $D = \mathbb{R} = (-\infty, \infty)$
   
   $E = [0, \infty)$

2. $f(x) = \frac{1}{x-1}$
   
   $D = \{ x \in \mathbb{R} : x \neq 1 \} = (-\infty, 1) \cup (1, \infty)$
   
   $E = (-\infty, 0) \cup (0, \infty)$
Visualizing a function

There are different ways to picture a function. One of them is an arrow diagram:

Each arrow connects an element of $D$ to an element of $E$. 
The most common way to picture a function is by drawing a graph.

**Definition:** A graph is the set of ordered pairs \( \{(x, f(x)) | x \in D \} \).

**Example:** Given \( f(x) = x^2 - 2x + 1 \), find \( f(6) \).

\[
f(6) = 6^2 - 2\cdot 6 + 1 = 25
\]
Example: Graph $f(x) = x + 2$

Example: Graph $f(x) = |x|$
When you look at the graph, how do you know you are looking at a function?

**Vertical Line Test:** A curve in the $xy$-plane is the graph of a function of $x$ if and only if no vertical line intersects the curve more than once.
Example: $x = y^2 - 1$

\[
y^2 = x + 1
\]

\[
y = \pm \sqrt{x + 1}
\]

\[
y = \sqrt{x + 1}
\]

\[
y = -\sqrt{x + 1}
\]

\(\text{not a function}\)

\(\text{functions}\)
Mathematical models: What kind of functions are there?

A mathematical model is a mathematical description (function or equation) of a real-world phenomenon.

Example: There is a strong positive linear relationship between husband's age and wife's age.

We can use a linear model to describe this relationship!
**Definition:** We say $y$ is a **linear function** of $x$ if $y = f(x) = mx + b$

- equation of a line, where

  - $m$ is the **slope** of the line, the amount by which $y$ changes when $x$ increases by one unit.
  - $b$ is the **y-intercept**, the value of $y$ when $x = 0$.

**Example:** $y = -0.5x + 1$
**Definition:** A function $f$ is a **polynomial function** if there are real numbers $a_0, a_1, ..., a_n$ such that $P(x) = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, for all $x$, $n$ is a nonnegative integer.

The numbers $a_0, a_1, ..., a_n$ are called **coefficients** of the polynomial. The highest power of $x$ with a nonzero coefficient is called the **degree** of the polynomial.

**Examples:**

1) A polynomial of degree 0 is a constant function $f(x) = c$
   e.g. $y = 3$

2) A polynomial of degree 1 is a linear function $f(x) = mx + b$. 
3) A polynomial of degree 2 is a quadratic function \( f(x) = ax^2 + bx + c \), e.g.

\[
y = x^2 - 2x + 1 = (x - 1)^2
\]

The graph is called a \textit{parabola}.

4) A polynomial of degree 3 is a cubic function \( f(x) = ax^3 + bx^2 + cx + d \), e.g. \( y = x^3 \)
Definition: If \( f(-x) = f(x) \) for every \( x \in D \), then \( f \) is called an **even function**. If \( f(-x) = -f(x) \) for every \( x \in D \), then \( f \) is called an **odd function**.

Example:

\( f(x) = x^2 \) is an even polynomial function.

\[
\begin{align*}
  f(-x) &= (-x)^2 = x^2 = f(x)
\end{align*}
\]

The graph of an even function is symmetric with respect to the \( y \)-axis.

\( f(x) = x^3 \) is an odd polynomial function.

\[
\begin{align*}
  f(-x) &= (-x)^3 = -x^3 = -f(x)
\end{align*}
\]

The graph of an odd function is symmetric about the origin.
What about $f(x) = x^2 - 2x + 1$?

\[
\begin{align*}
    f(-x) &= (-x)^2 - 2(-x) + 1 \\
         &= x^2 + 2x + 1 
\end{align*}
\]

neither odd nor even
Definition: A function $f$ is called **increasing** on an interval $I$ if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on $I$ if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I$$
Example: Given \( f(x) = -x^2 + 4x - 4 \), find the intervals where \( f(x) \) is increasing/decreasing.

\[
f'(x) = -(x^2 - 4x + 4) = -(x - 2)^2
\]

\( f(x) \) increases on \( (-\infty, 2) \)

\( f(x) \) decreases on \( (2, \infty) \)
**Definition**: A function of the form \( f(x) = x^a \), where \( a \) is a constant, is called a **power function**. We consider the following cases:

- If \( a = n \), where \( n \) is a positive integer, then \( f(x) = x^n \) is a **polynomial function**.
- If \( a = 1/n \), where \( n \) is a positive integer, then \( f(x) = \sqrt[n]{x} \) is a **root function**.

**Example**: \( y = \sqrt{x} \)
• If \( a = -1 \), then \( f(x) = x^{-1} = \frac{1}{x} \) is a **reciprocal function**.

The graph is called a *hyperbola* with the coordinate axes as its asymptotes.
Definition: A function $f$ is called a **rational function**, if it can be written as a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

$Q(x) \neq 0$

Example: $f(x) = \frac{x^2-x+2}{x-3}$
Definition: A function \( f \) is called an **algebraic function** if it is constructed by applying algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) to the polynomials.

Examples:

\[
\begin{align*}
f(x) &= \sqrt{x^2 + 2} & f(x) &= \frac{1-x}{x^2+1} \\
\end{align*}
\]

\[
f(x) = \sqrt{x^2 + 2} + \frac{1-x}{x^2 + 1}
\]
Trigonometric functions (review):

\[ f(x) = \sin x \]

\[ f(x) = \cos x \]
The remaining functions: cosecant, secant, and cotangent, are the reciprocal of the ones above.

\[ f(x) = \tan x = \frac{\sin x}{\cos x} \]

\[ D = \left\{ x \in \mathbb{R}, x \neq \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z} \right\} \]

\[ E = \mathbb{R} \quad \text{period} = \pi \]
Partial table of values for trigonometric functions:

<table>
<thead>
<tr>
<th>Angle $\theta$</th>
<th>Degrees</th>
<th>Radians</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
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</thead>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
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<tr>
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<td>$2\pi$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Identities

**Pythagorean Identities:**
\[
\sin^2 \theta + \cos^2 \theta = 1 \\
\tan^2 \theta + 1 = \sec^2 \theta \\
\cot^2 \theta + 1 = \csc^2 \theta
\]

**Sum or Difference of Two Angles:**
\[
\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi \\
\cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi \\
\tan(\theta \pm \phi) = \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}
\]

**Law of Cosines:**
\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

**Reduction Formulas:**
\[
\sin(-\theta) = -\sin \theta \\
\cos(-\theta) = \cos \theta \\
\tan(-\theta) = -\tan \theta
\]

**Half-Angle Formulas:**
\[
\sin^2 \frac{\theta}{2} = \frac{1}{2} (1 - \cos \theta) \\
\cos^2 \frac{\theta}{2} = \frac{1}{2} (1 + \cos \theta)
\]

**Double-Angle Formulas:**
\[
\sin 2\theta = 2 \sin \theta \cos \theta \\
\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \cos^2 \theta - \sin^2 \theta
\]

**Reciprocal Identities:**
\[
csc \theta = \frac{1}{\sin \theta} \\
\sec \theta = \frac{1}{\cos \theta} \\
\cot \theta = \frac{1}{\tan \theta}
\]

**Quotient Identities:**
\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \\
\cot \theta = \frac{\cos \theta}{\sin \theta}
\]
**Exponential functions**

**Definition:** The function of the form $f(x) = a^x$, where the *base* $a$ is a positive constant, is called an **exponential function**.

Let's recall what that means.
Laws of Exponents: If $a$ and $b$ are positive numbers and $x$ and $y$ are any real numbers, then

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

Example: Simplify $\sqrt[5]{a^{\sqrt[5]{b}}}^{\frac{1}{2}}$.

\[
\frac{\sqrt[5]{a^{\sqrt[5]{b}}}^{\frac{1}{2}}}{\sqrt[5]{ab}} = a^{\frac{1}{2}} \left(b^{\frac{1}{5}}\right)^{\frac{1}{2}} = a^{\frac{5}{2}} b^{\frac{1}{5}} = a^{\frac{5}{2}} \left(b^{\frac{1}{5}}\right) = a^{\frac{5}{10}} b^{\frac{1}{10}}
\]

\[
= a^{\frac{1}{2}} b^{\frac{1}{10}} - \frac{1}{2}
\]

\[
= a^{\frac{1}{2}} b^{\frac{1}{10}} - \frac{1}{2}
\]
The number $e$

\[ y = mx + l \]
\[ m = 1 \]

\[(0,1)\]

\[ y = e^x \]

\[
e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.
\]

\[ \approx 2.71828 \]
How can we get new functions from the ones we know?

**Transformations of functions**

**Vertical and Horizontal Shifts**: Suppose $c > 0$. To obtain the graph of

- $y = f(x) \pm c$, shift the graph of $y = f(x)$ a distance $c$ units upward/downward
- $y = f(x \pm c)$, shift the graph of $y = f(x)$ a distance $c$ units to the left/right

**Example**: $f(x) = (x - 2)^2 + 1$

![Graph showing transformations of functions with a point labeled (2,1) as the vertex of the parabola. The graph illustrates the vertical and horizontal shifts of the function.](image)
Vertical and Horizontal Stretching: Suppose $c > 0$. To obtain the graph of

- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of $c$
- $y = \frac{1}{c}f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of $c$
- $y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of $c$
- $y = f\left(\frac{x}{c}\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of $c$

Example: $y = \sin 2x$
Reflecting: To obtain the graph of

- $y = -f(x)$, reflect the graph of $y = f(x)$ about the $x$-axis
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the $y$-axis

Example: $y = 2^{-x-1} = \frac{1}{2^{x+1}}$
Combinations of functions

\[(f \pm g)(x) = f(x) \pm g(x) \text{ (sum/difference)}\]

\[(fg)(x) = f(x)g(x) \text{ (product)}\]

\[\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0 \text{ (quotient)}\]

\[(f \circ g)(x) = f(g(x)) \text{ (composite function)}\]

Example: If \(f(x) = e^x\) and \(g(x) = \sin^2 x\), find \(f \circ g, g \circ f, \) and \(g + f \circ f\).

\[f \circ g = f(g(x)) = f(\sin^2 x) = e^{\sin^2 x}\]

\[g \circ f = g(f(x)) = g(e^x) = \sin^2(e^x)\]

\[g + f \circ f = \sin^2 x + e^x\]

What about \(f \circ f \circ f\)?

\[= f(f(f(x))) = e^{e^x}\]
Inverse functions

**Definition:** A function $f$ is called a one-to-one function if it never takes on the same value twice, i.e.

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

**Example:** $y = x^2$. Is it one-to-one?
How to check?

**Horizontal line test:** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

*One-to-one Function: Yes*  

*One-to-one Function: No*
**Definition:** Let $f$ be one-to-one function with domain $A$ and range $B$. Then its **inverse function** $f^{-1}$ has domain $B$ and range $A$ and is defined by

$$f^{-1}(y) = x \iff f(x) = y \text{ for any } y \in B$$

**Note:** $f^{-1}(x) \neq \frac{1}{f(x)}$

**Example:** Given that $f(x)$ is one-to-one, and $f(0) = -1$, $f(2) = 0$, $f(3) = 2$. Find $f^{-1}(-1)$, $f^{-1}(0)$, and $f(f^{-1}(2))$. 

\[
f^{-1}(-1) = x \\
f(x) = -1 \implies x = 0, \text{ so } f^{-1}(-1) = 0 \\
f^{-1}(0) = 2 \\
f(f^{-1}(2)) = f(3) = 2
\]
Note: Inverse functions have the unique property that, when composed with their original functions, both functions cancel out. Mathematically, this means that

\[ f^{-1}(f(x)) = x, \quad x \in A \]
\[ f(f^{-1}(x)) = x, \quad x \in B \]

Since functions and inverse functions contain the same numbers in their ordered pair, just in reverse order, their graphs will be reflections of one another across the line \( y = x \):
Example: $f(x) = x^3$

$f^{-1}(x) = \sqrt[3]{x}$
How to find the inverse function?

To find the inverse function for a one-to-one function, follow these steps:

1. Rewrite the function using \( y \) instead of \( f(x) \).
2. Solve the equation for \( x \) in term of \( y \).
3. Switch the \( x \) and \( y \) variables
4. The resulting equation is \( y = f^{-1}(x) \)
5. Make sure that your resulting inverse function is one-to-one. If it isn't, restrict the domain to pass the horizontal line test.
Example: Given \( f(x) = \sqrt{x + 3} \), find \( f^{-1}(x) \).

1. \( y = \sqrt{x + 3} \)
2. \( y^2 = x + 3 \)
   \( x = y^2 - 3 \)
3. \( y = x^2 + 3 \)
4. \( f^{-1}(x) = x^2 + 3 \rightarrow \text{not one-to-one} \)

Note: \( x \geq 0 \) for \( f^{-1}(x) \). Without this restriction, \( f^{-1}(x) \) would not pass the horizontal line test. It obviously must be one-to-one, since it must possess an inverse of \( f(x) \). You should use that portion of the graph because it is the reflection of \( f(x) \) across the line \( y = x \), unlike the portion on \( x < 0 \).
Examples of inverse functions you need to know

- Logarithmic functions

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is one-to-one, so it has an inverse function $f^{-1}$ called the **logarithmic function with base $a$**.

Notation: $\log_a$

Thus,

$$f^{-1}(x) = \log_a x = y \iff f(y) = a^y = x$$

Cancellation property:

$$f^{-1}(f(x)) = \log_a (a^x) = x, \quad x \in \mathbb{R}$$

$$f(f^{-1}(x)) = a^{\log_a x} = x, \quad x \in \mathbb{R}$$
Laws of logarithms: Given $x,y \in \mathbb{Z}^+$ (positive integers)

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a \frac{x}{y} = \log_a x - \log_a y$

3. $\log_a x^r = r \log_a x$, $r \in \mathbb{R}$

Note: $\log_a a = 1$

Example: Evaluate $\log_2 5 - \log_2 40 - \log_2 1$

\[
\begin{align*}
= \log_2 \frac{5}{40} & - 0 \\
= \log_2 \frac{1}{8} & = \log_2 2^{-3} \\
= -3 \log_2 2 & = -3 \cdot 1 = -3
\end{align*}
\]
**Definition:** The logarithm with base $e$ is called the **natural logarithm**.

**Notation:** $\log_e x = \ln x$

So,

\[
\ln x = y \iff e^y = x
\]

\[
\ln e^x = x, \quad x \in \mathbb{R}
\]

\[
e^{\ln x} = x, \quad x > 0
\]

\[
\ln e = 1
\]

**Example:** Solve $e^{3-x} = 7$

\[
\ln e^{3-x} = \ln 7
\]

\[
3-x = \ln 7
\]

\[
x = 3 - \ln 7
\]
Change of base formula:

\[ \log_a x = \frac{\ln x}{\ln a}, \quad a > 0, \ a \neq 1 \]

Example: Evaluate \( \log_8 3 \)

\[
\begin{align*}
\log_8 3 &= \frac{\ln 3}{\ln 8} \\
&= \frac{0.528}{0.528} \\
&= 0.528
\end{align*}
\]

calculator
• **Inverse trigonometric functions**

Inverse sine function or arcsine function: \( \sin^{-1} x \)

\[ y = \arcsin x \]

\[ y = x \]

\[ y = \sin x \]

\[ \text{Domain: } [-1, 1] \]

\[ \text{Range: } \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \]

\( y = \sin x \) is not one-to-one, but for \(-\pi/2 \leq x \leq \pi/2\) it is.
So we have

\[ \sin^{-1} x = y \iff \sin y = x \text{ and } -\pi/2 \leq y \leq \pi/2 \]

\[ \sin^{-1} (\sin x) = x, \quad -\pi/2 \leq x \leq \pi/2 \]

\[ \sin (\sin^{-1} x) = x, \quad -1 \leq x \leq 1 \]

Example: Evaluate

(a) \( \sin^{-1} 1/2 \)

\[ \sin (\sin^{-1} 1/2) = \sin x \]

\[ \frac{1}{2} = \sin x \Rightarrow x = \frac{\pi}{6} \]

(b) \( \cos \sin^{-1} \frac{1}{\sqrt{2}} \)

\[ \sin^{-1} \frac{1}{\sqrt{2}} = x \]

\[ \frac{1}{\sqrt{2}} = \sin x \Rightarrow x = \frac{\pi}{4} \]
Similarly we can define inverse functions for other trigonometric functions:

- $f(x) = \cos^{-1}(x)$
  - Domain: $[-1,1]$
  - Range: $[0,\pi]$  

- $f(x) = \arccos(x)$
  - Domain: $[-1,1]$
  - Range: $[0,\pi]$  

- $f(x) = \tan^{-1}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

- $f(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
\[ y = \cot^{-1} x, x \in \mathbb{R} \iff \cot y = x, \quad y \in (0, \pi) \]

\[ y = \sec^{-1} x, |x| \geq 1 \iff \sec y = x, y \in [0, \pi/2) \cup [\pi, 3\pi/2) \]

\[ y = \csc^{-1} x, |x| \geq 1 \iff \csc y = x, y \in (0, \pi/2] \cup (\pi, 3\pi/2] \]
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General solutions

Note: trigonometric functions are periodic.

This periodicity is reflected in the general inverses:

\[
\sin(y) = x \iff y = \arcsin(x) + 2k\pi \text{ or } y = \pi - \arcsin(x) + 2k\pi, \ k \in \mathbb{Z}
\]
or

\[
\sin(y) = x \iff y = (-1)^k \arcsin(x) + k\pi
\]

\[
\cos(y) = x \iff y = \arccos(x) + 2k\pi \text{ or } y = 2\pi - \arccos(x) + 2k\pi
\]
or

\[
\cos(y) = x \iff y = \pm \arccos(x) + 2k\pi
\]

\[
\tan(y) = x \iff y = \arctan(x) + k\pi
\]

\[
\cot(y) = x \iff y = \arccot(x) + k\pi
\]

\[
\sec(y) = x \iff y = \arcsec(x) + 2k\pi \text{ or } y = 2\pi - \arcsec(x) + 2k\pi
\]

\[
\csc(y) = x \iff y = \arccsc(x) + 2k\pi \text{ or } y = \pi - \arccsc(x) + 2k\pi
\]
Example: Solve equation \( 2 \sin 2x + 1 = 0 \)

\[
\sinh 2x = -\frac{1}{2}
\]

\[\Theta = 2x\]

\[\Theta = 2x = \frac{7\pi}{6} + 2k\pi, \; k \in \mathbb{Z}\]

\[\frac{11\pi}{6} + 2k\pi, \; k \in \mathbb{Z}\]

\[x = \frac{7\pi}{12} + k\pi, \; k \in \mathbb{Z}\]

\[\frac{11\pi}{12} + k\pi, \; k \in \mathbb{Z}\]