On Bounding the Union Probability

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Abstract—We present new results on bounding the probability of a finite union of events, \( P \left( \bigcup_{i=1}^{N} A_i \right) \) for a fixed positive integer \( N \), using partial information on the events joint probabilities. We first consider bounds that are established in terms of \( \{P(A_i)\} \) and \( \{\sum_j c_j P(A_i \cap A_j)\} \) where \( c_1, \ldots, c_N \) are given weights. We derive a new class of lower bounds of at most pseudo-polynomial computational complexity. This class of lower bounds generalizes the recent bounds in [1, 2] and can be tighter in some cases than the Gallot-Kounias [3]–[5] and Prekopa-Gao [6] bounds which require more information on the events probabilities. We next consider bounds that fully exploit knowledge of \( \{P(A_i)\} \) and \( \{P(A_i \cap A_j)\} \). We establish new numerical lower/upper bounds on the union probability by solving a linear programming problem with \( N^2 + N + 2 \) variables. These bounds coincide with the optimal lower/upper bounds when \( N \leq 7 \) and are guaranteed to be sharper than the optimal lower/upper bounds of [1, 2] that use \( \{P(A_i)\} \) and \( \{\sum_j c_j P(A_i \cap A_j)\} \).

Index Terms—Union probability, upper and lower bounds, linear programming, probability of error analysis, communication systems.

I. INTRODUCTION

Lower/upper bounds on the union probability \( P \left( \bigcup_{i=1}^{N} A_i \right) \) in terms of the individual event probabilities \( P(A_i) \)'s and the pairwise event probabilities \( P(A_i \cap A_j) \)'s were actively investigated in the recent past. The optimal bounds can be obtained numerically by solving linear programming (LP) problems with \( 2^N \) variables [6], [7]. Since the number of variables is exponential in the number of events, \( N \), some suboptimal but numerically efficient bounds were proposed, such as the bounds in [8] that employ the dual basic feasible solutions to reduce the complexity of the LP problem, and the algorithmic Bonferroni-type lower/upper bounds in [9], [10].

Among the established analytical bounds is the Kuai-Alajaji-Takahara lower bound (for convenience, hereafter referred to as the KAT bound) [11] that was shown to be better than the Dawson-Sankoff (DS) [12] and the D. de Caen (DC) [13] bounds. Noting that the KAT bound is expressed in terms of \( \{P(A_i)\} \) and only the sums of the pairwise event probabilities, i.e., \( \{\sum_{j \neq i} c_j P(A_i \cap A_j)\} \), in order to fully exploit all pairwise event probabilities, it is observed in [14]–[16] that the analytical bounds can be further improved algorithmically by optimizing over subsets. Furthermore, in [6], the KAT bound is extended by using additional partial information such as the sums of joint probabilities of three events, i.e., \( \{\sum_{j \neq i} c_j P(A_i \cap A_j \cap A_k), i = 1, \ldots, N\} \). Recently, using the same partial information as the KAT bound, i.e., \( \{P(A_i)\} \) and \( \{\sum_{j \neq i, l} c_j P(A_i \cap A_j \cap A_l)\} \), the optimal lower/upper bound as well as a new analytical bound which is sharper than the KAT bound were developed in [11, 2].

In this paper, we first establish a new class of lower bounds on \( P \left( \bigcup_{i=1}^{N} A_i \right) \) using \( \{P(A_i)\} \) and \( \{\sum_j c_j P(A_i \cap A_j)\} \) for a given weight or parameter vector \( c = (c_1, \ldots, c_N)^T \). These lower bounds are shown to have at most pseudo-polynomial computational complexity and to be sharper in certain cases than the existing Gallot-Kounias (GK) [3]–[5] and Prekopa-Gao (PG) [6] bounds, although the later bounds employ more information on the events joint probabilities. Furthermore, for bounds on \( P \left( \bigcup_{i=1}^{N} A_i \right) \) that fully exploit knowledge of \( \{P(A_i)\} \) and \( \{P(A_i \cap A_j)\} \), a new numerical lower/upper bound is proposed by solving an LP problem with \( (N-1)^2 + N + 1 \) variables. This numerical lower/upper bound is proven to be an optimal lower/upper bound when \( N \leq 7 \) and to be always better than the optimal lower/upper bound which uses \( \{P(A_i)\} \) and \( \{\sum_j P(A_i \cap A_j)\} \). Finally, we should note that these general union probability bounds can be applied to effectively estimate and analyze the error performance of a variety of coded or uncoded communication systems (e.g., see [2], [9], [10], [14], [17], [22]).

II. NEW BOUNDS USING \( \{P(A_i)\} \) AND \( \{\sum_j c_j P(A_i \cap A_j)\} \)

For simplicity, and without loss of generality, we assume the events \( \{A_1, \ldots, A_N\} \) are in a finite probability space \( (\Omega, \mathcal{F}, P) \), where \( N \) is a fixed positive integer. Let \( \mathcal{B} \) denote the collection of all non-empty subsets of \( \{1, 2, \ldots, N\} \). Given \( B \in \mathcal{B} \), we let \( \omega_B \) denote the atom in the union \( \bigcup_{i \in B} A_i \) such that for all \( i = 1, \ldots, N \), \( \omega_B \in A_i \) if \( i \in B \) and \( \omega_B \notin A_i \) if \( i \notin B \) (note that some of these “atoms” may be the empty set). For ease of notation, for a singleton \( \omega \in \Omega \), we denote \( P(\{\omega\}) \) by \( p(\omega) \) and \( p(\omega_B) \) by \( p_B \). Since \( \{\omega_B : i \in B\} \) is the collection of all the atoms in \( A_i \), we have \( P(A_i) = \sum_{\omega \in A_i} p(\omega) = \sum_{B \in \mathcal{B} : i \in B} p_B \), and

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\[ P \left( \bigcup_{i=1}^{N} A_i \right) = \sum_{B \in \mathcal{B}} p_B. \] (1)

Suppose there are \( N \) functions \( f_i(B), i = 1, \ldots, N \) such that \( \sum_{i=1}^{N} f_i(B) = 1 \) for any \( B \in \mathcal{B} \) (i.e., for any atom \( \omega_B \)). If we further assume that \( f_i(B) = 0 \) if \( i \notin B \) (i.e., \( \omega_B \notin A_i \)), we can write

\[ P \left( \bigcup_{i=1}^{N} A_i \right) = \sum_{B \in \mathcal{B}} \left( \sum_{i=1}^{N} f_i(B) \right) p_B = \sum_{i=1}^{N} \sum_{B \in \mathcal{B}, i \in B} f_i(B)p_B. \] (2)

Note that if we define

\[ f_i(B) = \begin{cases} \frac{1}{|B|} = \frac{1}{\deg(\omega_B)} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases} \] (3)

where the degree of \( \omega, \deg(\omega) \), is the number of \( A_i \)'s that contain \( \omega \), then \( \sum_{i=1}^{N} f_i(B) = 1 \) is satisfied and (2) becomes

\[ P \left( \bigcup_{i=1}^{N} A_i \right) = \sum_{i=1}^{N} \sum_{\omega \in A_i} \frac{p(\omega)}{\deg(\omega)}. \] (4)

Note that many of the existing bounds, such as the DC bound [13] and KAT bound [11] and the bounds in [11] [2], are based on (4).

In the following lemma, we propose a generalized expression of (4). To the best of our knowledge this lemma is novel.

**Lemma 1:** Suppose \( \{\omega_B, B \in \mathcal{B}\} \) are all the \( 2^N - 1 \) atoms in \( \bigcup_{i} A_i \). If \( c = (c_1, \ldots, c_N)^T \in \mathbb{R}^N \) satisfies

\[ \sum_{k \in B} c_k \neq 0, \quad \text{for all } B \in \mathcal{B}, \] then we have

\[ P \left( \bigcup_{i=1}^{N} A_i \right) = \sum_{i=1}^{N} \sum_{B \in \mathcal{B}, i \in B} \frac{c_i p_B}{\sum_{k \in B} c_k} = \sum_{i=1}^{N} \sum_{\omega \in A_i} \frac{c_i p(\omega)}{\sum_{k \in A_i} c_k}. \] (5)

**Proof:** If we define

\[ f_i(B) = \begin{cases} \frac{c_i}{\sum_{k \in B} c_k} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases} \] (7)

where the parameter vector \( c = (c_1, c_2, \ldots, c_N)^T \) satisfies \( \sum_{k \in B} c_k \neq 0 \) for all \( B \in \mathcal{B} \) (therefore \( c_i \neq 0, i = 1, \ldots, N \)), then \( \sum_{i=1}^{N} f_i(\omega) = 1 \) holds and we can get (6) from (2).

Note that (6) holds for any \( c \) that satisfies (5) and is clearly a generalized expression of (4).

**A. Relation to the Cohen-Merhav bound [19]**

Let \( m_i(\omega) \) be non-negative functions. Then by the Cauchy-Schwarz inequality,

\[ \left[ \sum_{B \in \mathcal{B}} f_i(B)p_B \right] \left[ \sum_{B \in \mathcal{B}} \frac{p_B}{f_i(B)} m_i^2(\omega_B) \right] \geq \left[ \sum_{B \in \mathcal{B}} p_B m_i(\omega_B) \right]^2. \] (8)

Thus, using (2), we have

\[ P \left( \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i=1}^{N} \sum_{B \in \mathcal{B}} \frac{p_B m_i(\omega_B)^2}{\sum_{k \in B} c_k P(A_i \cap A_k)}. \] (9)

If we define \( f_i(B) \) by (3), then (9) reduces to

\[ P \left( \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i=1}^{N} \sum_{\omega \in A_i} \frac{p(\omega) m_i(\omega)^2}{\sum_{k \in A_i} c_k P(\omega)}. \] (10)

which is the Cohen-Merhav lower bound in [19, Theorem 2.1]; note that equality in (10) holds when \( m_i(\omega) = \frac{1}{\deg(\omega)} \) (i.e., \( m_i(\omega_B) = \frac{1}{|B|} \)).

**B. Relation to the GK bound [3], [4]**

In this subsection, we assume that the elements of \( c \) are positive, i.e., \( c \in \mathbb{R}^N_+ \), and connect the GK bound [3] [4] with (6). The GK bound was recently revisited in [5] where it is reformulated as

\[ \ell_{GK} = \max_{c \in \mathbb{R}^N_+} \frac{\sum_{i} c_i P(A_i)^2}{\sum_{B \in \mathcal{B}} \sum_{k \in B} c_k P(A \cap A_k)}, \] (11)

and the optimal \( c \) for (11), denoted by \( \hat{c} \), can be computed by

\[ \hat{c} = \Sigma^{-1} \alpha, \] (12)

where \( \alpha = (P(A_1), \ldots, P(A_N))^T \) and \( \Sigma \) is the \( N \times N \) matrix whose \( (i,j) \)-th element is \( P(A_i \cap A_j) \).

First, consider \( c \in \mathbb{R}^N_+ \) fixed. Then, by the Cauchy-Schwarz inequality, we have

\[ \left[ \sum_{B \in \mathcal{B}} \frac{c_i p_B}{\sum_{k \in B} c_k} \right] \left[ \sum_{B \in \mathcal{B}} \left( \frac{\sum_{k \in B} c_k P(A_i \cap A_k)}{c_i} \right) p_B \right] \geq P(A_i)^2. \] (13)

Note that

\[ \sum_{B \in \mathcal{B}} \left( \frac{\sum_{k \in B} c_k P(A_i \cap A_k)}{c_i} \right) p_B = \frac{1}{c_i} \sum_{k \in \mathcal{B}} c_k P(A_i \cap A_k). \] (14)

Then for all \( i \),

\[ \sum_{B \in \mathcal{B}} \frac{c_i p_B}{\sum_{k \in B} c_k} \geq \frac{c_i^2 P(A_i)^2}{\sum_{k \in \mathcal{B}} c_k P(A_i \cap A_k)}. \] (15)

By summing (15) over \( i \), we get another new lower bound:

\[ P \left( \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i=1}^{N} \frac{c_i^2 P(A_i)^2}{\sum_{k \in \mathcal{B}} c_k P(A_i \cap A_k)}. \] (16)

Note that we can use Cauchy-Schwarz Inequality again:

\[ \left[ \sum_{i=1}^{N} \frac{c_i^2 P(A_i)^2}{\sum_{k \in \mathcal{B}} c_k P(A_i \cap A_k)} \right] \left[ \sum_{i=1}^{N} \frac{\sum_{k \in \mathcal{B}} c_k P(A_i \cap A_k)}{c_i} \right] \geq \left[ \sum_{i=1}^{N} c_i P(A_i \cap A_k) \right]^2. \] (17)
Since the above inequality holds for any positive $c$, we have
\[
P\left(\bigcup_{i=1}^{N} A_i\right) \geq \max_{c \in \mathbb{R}_+^N} \sum_{i=1}^{N} c_i^2 P(A_i)^2 \geq \max_{c \in \mathbb{R}_+^N} \frac{\left(\sum_{i=1}^{N} c_i P(A_i)\right)^2}{\sum_{i=1}^{N} c_i P(A_i)}.
\] (18)

Note that the lower bounds in (18) are weaker than the GK bound (11), however, if the optimal $c$ of (11), $\hat{c}$, happen to satisfy $\hat{c} \in \mathbb{R}_+^N$, then the bounds in (18) coincide with the GK bound (11).

C. New Class of Lower Bounds

We only consider $c \in \mathbb{R}_+^N$ in this subsection. A new class of lower bounds is given in the following theorem.

**Theorem 1:** Defining $\mathcal{B}^- = \mathcal{B} \setminus \{1, \ldots, N\}$, $\tilde{\gamma}_i := \sum_{k \in B_i} c_k P(A_i)\cap A_k$, $\tilde{\alpha}_i := P(A_i)$ and
\[
\tilde{\delta} := \max_i \left[ \frac{\tilde{\gamma}_i - (\sum_{k} c_k - \min_k c_k) \tilde{\alpha}_i}{\min_k c_k} \right],
\] (19)

where $c \in \mathbb{R}_+^N$, a class of lower bounds is given by
\[
P\left(\bigcup_{i=1}^{N} A_i\right) \geq \tilde{\delta} + \sum_{i=1}^{N} \ell'(c, \tilde{\delta}),
\] (20)

where
\[
\ell'(c, x) = \left[ P(A_i) - x \right] \left( \frac{c_i}{\sum_{k \in B_1} c_k} + \frac{c_i}{\sum_{k \in B_2} c_k} - \frac{c_i \sum_{k \in B_1} c_k P(A_i) \cap A_k - x}{P(A_i) - x} \left( \frac{\sum_{k \in B_1} c_k}{\sum_{k \in B_2} c_k} \right) \right),
\] (21)

and
\[
B_1^{(i)} = \arg \max_{\{B \in \mathcal{B}^- : i \notin B\}} \frac{\sum_{k \in B} c_k}{c_i},
\]
\[
\text{s.t.} \quad \sum_{k \in B} c_k \leq \sum_{k \in B} c_k \left[ P(A_i) \cap A_k - x \right],
\]
\[
B_2^{(i)} = \arg \min_{\{B \in \mathcal{B}^- : i \notin B\}} \frac{\sum_{k \in B} c_k}{c_i},
\]
\[
\text{s.t.} \quad \sum_{k \in B} c_k \geq \sum_{k \in B} c_k \left[ P(A_i) \cap A_k - x \right].
\] (22)

**Proof:** Let $x = p(1, 2, \ldots, N)$ and consider $\sum_i \ell'(c, x) + x$ as a new lower bound where where $\ell'(c, x)$ equals to the objective value of the problem
\[
\min_{\{p_B : i \in B, B \in \mathcal{B}^-\}} \sum_{B \in \mathcal{B}^-} \sum_{B \in \mathcal{B}^-} \frac{c_i p_B}{c_i P_B} \sum_{B: B \notin B} p_B = P(A_i) - x,
\]
\[
\text{s.t.} \quad \sum_{B: B \notin B} p_B = P(A_i) - x,
\]
\[
\sum_{B: B \notin B} \left( \frac{\sum_{k \in B} c_k}{c_i} \right) p_B = \frac{1}{c_i} \sum_{k \in A_i} c_k \left[ P(A_i) \cap A_k - x \right],
\]
\[
p_B \geq 0, \text{ for all } B \in \mathcal{B}^- \text{ such that } i \in B.
\] (23)

The solution of (23) exists if and only if
\[
\min_k c_k \leq \frac{\tilde{\gamma}_i - (\sum_{k} c_k - \min_k c_k) \tilde{\alpha}_i}{\min_k c_k} \leq \sum_k c_k - \min_k c_k.
\] (24)

Therefore, the new lower bound can be written as
\[
\min_{\tilde{\delta}} \left[ x + \sum_{i=1}^{N} \ell'(c, x) \right] \quad \text{s.t.} \quad \left[ \tilde{\gamma}_i - (\sum_{k} c_k - \min_k c_k) \tilde{\alpha}_i \right] \leq x \leq \left[ \tilde{\gamma}_i - (\min_k c_k) \tilde{\alpha}_i \right], \forall i.
\] (25)

We can prove that the objective function of (25) is non-decreasing with $x$. Therefore, defining $\tilde{\delta}$ as in (19), the new lower bound can be written as (20) where $\ell'(c, \tilde{\delta})$ can be obtained by solving (23), which is given in (21).

**Remark 1:** Note that the problems in (22) are exactly the 0/1 knapsack problem with mass equals to value [23], which can be computed in pseudo-polynomial time, and can be arbitrarily closely approximated by an algorithm running in polynomial time [23].

**Remark 2:** It can readily be shown that if $c = \kappa 1$ for any non-zero constant $\kappa$ with 1 being the all-one vector of length $N$, the new lower bound reduces to the analytical lower bound in [1], [2], which is sharper than the KT bound. It can also be shown that if the optimal $\tilde{c}$ of the GK bound satisfies $\tilde{c} \in \mathbb{R}_+^N$, then the new lower bound is sharper than the GK bound.

III. NEW BOUNDS USING $\{P(A_i)\}$ AND $\{P(A_i \cap A_j)\}$

In this section, we derive new numerical lower/upper bounds for $P\left(\bigcup_{i=1}^{N} A_i\right)$ using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$. First, consider the $p_B$'s in (1) as variables. Then the following (exclusive) LP problem with $2^N$ variables gives the optimal lower/upper bound established using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\})$:
\[
\min_{\{p_B : B \in \mathcal{B}\}} \sum_{B \in \mathcal{B}} p_B \quad \text{s.t.} \quad \sum_{i,j \in B, B \in \mathcal{B}} p_B = P(A_i \cap A_j), \quad i, j \in \{1, \ldots, N\},
\]
\[
p_B \geq 0, B \in \mathcal{B}.
\] (26)

The optimality of (26) can be easily proved by showing its achievability: for each $p_B$, construct an atom $\omega_B$ such that $p(\omega_B) = p_B$ and let $\omega_B \in A_i, \forall i \in B$. However, the computational complexity of the optimal lower/upper bound...
in (26) is exponential. Next, we consider a relaxed problem of (26), which is given in the following:

$$\min_{\{p_B, B \in \mathcal{B}\}} \max_{\{p_B, B \in \mathcal{B}\}} \sum_{B \in \mathcal{B}} p_B,$$

s.t.

$$\sum_{i,j \in B, B \in \mathcal{B}} p_B = P(A_i \cap A_j), \quad i, j \in \{1, \ldots, N\},$$

$$\sum_{B_i,j \in B, |B| = k} p_B \geq 0, \quad \sum_{B_i,j \notin B, |B| = k} p_B \geq 0,$$

$$\sum_{B_i \in B, |B| = k} p_B \geq 0, \quad \sum_{B_i,j \notin B, |B| = k} p_B \geq 0,$$

$$\sum_{B_i \in B, |B| = k} p_B \geq 0, \quad \sum_{B_i,j \notin B, |B| = k} p_B \geq 0,$$

$$\sum_{B_i,j \in B, |B| = k} p_B \geq 0, \quad \sum_{B_i,j \notin B, |B| = k} p_B \geq 0,$$

$$\forall i, j, l, k \in \{1, \ldots, N\}.$$  

(27)

Since the solution of (27) is a lower/upper bound for the union probability $P\left(\bigcup_{i=1}^{N} A_i\right)$, we next show that the solution of (27) can be obtained by solving an LP problem with \((N-1)^2 + N + 3\) variables, which coincides with the optimal lower/upper bounds when \(N \leq 7\). The main results are in the following.

Lemma 2: The solution of problem (27) coincides with the optimal lower/upper bound in (26) when \(N \leq 7\).

Lemma 3: The problem (27) shares the same solution with the following LP:

$$\min_{\{p_B, B \in \mathcal{B}\}} \max_{\{p_B, B \in \mathcal{B}\}} \sum_{B \in \mathcal{B}} p_B,$$

s.t.

$$\sum_{i,j \in B, B \in \mathcal{B}} p_B = P(A_i \cap A_j), \quad i, j \in \{1, \ldots, N\},$$

$$\sum_{B_i,j \in B, |B| = k} p_B \geq 0, \quad \sum_{B_i,j \notin B, |B| = k} p_B \geq 0,$$

$$\sum_{B_i \in B, |B| = k} p_B \geq 0, \quad \sum_{B_i,j \notin B, |B| = k} p_B \geq 0,$$

$$\sum_{B_i,j \in B, |B| = k} p_B \geq 0, \quad \sum_{B_i,j \notin B, |B| = k} p_B \geq 0,$$

$$\sum_{B_i,j \in B, |B| = k} p_B \geq 0, \quad \sum_{B_i,j \notin B, |B| = k} p_B \geq 0,$$

$$\forall i, j, l, k \in \{1, \ldots, N\}.$$  

(28)

Theorem 2: Defining $a_{ij}(k) = \sum_{i,j \in B, |B| = k} p_B$, the LP problem (28) can be reformulated as an LP of \(\{a_{ij}(k)\}\) (i.e., \(N^3\) variables). The number of variables can hence be reduced from \(N^3\) to \(\frac{(N-1)^2 + N + 3}{2}\).

Proof: Define $a(k) = \sum_{|B| = k} p_B$ and $a_i(k) = \sum_{i \in B, |B| = k} p_B$, then it can be readily shown that $a(k) = \sum_{i=1}^{N} \frac{a_i(k)}{k}$ and $a_i(k) = \sum_{j=1}^{N} \frac{a_{ij}(k)}{k}$. Therefore, both $a(k)$ and $a_i(k)$ are linear functions of $\{a_{ij}(k)\}$.

We next demonstrate that the number of variables can be reduced from \(N^3\) to \(\frac{(N-1)^2 + N + 3}{2}\). Note that according to the definition of $a_{ij}(k)$, we have: i) $a_{ij}(1) = P\left(\{x \in A_i \cap A_j, \deg(x) = 1\}\right) = 0, \forall i \neq j$; ii) $a_{ij}(k) = a_{ij}(k)$; iii) $a_{ij}(N) = P\left(\bigcap_{i=1}^{N} A_i\right)$ for any $i$ and $j$. Therefore, the number of variables for different values of $k$ can be reduced to

$$\begin{cases} 
N(N-1) & \text{if } k = 1 \\
\frac{N-1}{2} & \text{if } k = 2, \ldots, N-1 \\
1 & \text{if } k = N 
\end{cases}$$

(29)

Thus, the total number of variables is $N + \frac{(N-1)(N-2)}{2} + 1$.

Now it is suffices to show that the objective function and all the constraints in (28) can be written as functions of $a_{ij}(k)$ so that all $\{p_B\}$ can be replaced using $a_{ij}(k)$. In the following, we directly give the results, which one can easily verify.

The objective function and the first constraint of (28) can be written as

$$\sum_{k} \sum_{i,j} \frac{a_{ij}(k)}{k^2} = \sum_{B \in \mathcal{B}} p_B,$$

(30)

Finally, for all $i, j, l, k \in \{1, \ldots, N\}$, the other constraints of (28) as functions of $\{p_B\}$ can be written as functions of $\{a_{ij}(k)\}$ as follows:

$$a_{ij}(k) = \sum_{B_i,j \in B, |B| = k} p_B + \sum_{B_i,j \notin B, |B| = k} p_B,$$

(31)

$$a(k) - a_i(k) - a_j(k) + a_{ij}(k) = \sum_{B \in \mathcal{B}} p_B + \sum_{B \notin \mathcal{B}} p_B,$$

$$a(k) - a_i(k) - a_j(k) + a_{ij}(k) + a_{il}(k) + a_{jl}(k) = \sum_{B \in \mathcal{B}} p_B + \sum_{B \notin \mathcal{B}} p_B,$$

$$a_{ij}(k) - a_{il}(k) - a_{jl}(k) = \sum_{B \in \mathcal{B}} p_B + \sum_{B \notin \mathcal{B}} p_B,$$

$$a_i(k) - a_{ij}(k) = \sum_{B \in \mathcal{B}} p_B + \sum_{B \notin \mathcal{B}} p_B.$$  

Therefore, the lower/upper bounds of (27) can be solved by an LP with \(\frac{(N-1)^2 + N + 3}{2}\) variables.

Remark 3: According to Lemma 2, the new numerical lower/upper bound coincides with the optimal lower/upper bounds in (26) when $N \leq 7$. Furthermore, we can show that the new numerical lower/upper bounds are sharper than the numerical bounds in [1], [2], which have been proved to be the optimal lower/upper bounds in terms of $\{P(A_i)\}$ and $\{\sum_{j} P(A_i \cap A_j)\}$.

IV. NUMERICAL EXAMPLES

Due to the space limitation, we only present lower bounds in this section. The same eight systems as in [1] are used and the corresponding results are shown in Table I. For comparison, we include bounds that utilize $\{P(A_i)\}$ and...


\[ \sum_{i} P(A_i \cap A_j), i = 1, \ldots, N, \] such as the KAT bound [11], the analytical bound in [1], [2], and the numerical optimal bound in this class [1], [2]. We also include the GK bound [3], [4] and the stepwise bound [9], which fully exploit \( \{P(A_i)\} \) and \( \{P(A_i \cap A_j)\} \). The PG lower bound [6], which extends the KAT bound by using \( \{P(A_i)\} \), \( \{\sum_j P(A_i \cap A_j)\} \) and \( \{\sum_{j,j} P(A_i \cap A_j)\} \), is also investigated in the examples. The Cohen-Merhav bound (10) [19] is not included since it is not clear how to choose the function \( m_1(\omega) \) in our examples. For the proposed bound (20) we consider two cases for choosing \( c \). The first choice for \( c \), denoted by \( \tilde{c}^+ \), has components \( \tilde{c}_i^+ = \max(\tilde{c}_i, \epsilon) \) with \( \tilde{c} \) given in (12) and \( \epsilon > 0 \) close to zero. Therefore, if \( \tilde{c} \in \mathbb{R}^N_+ \) then \( \tilde{c} \) is sharper than the GK bound. If \( \tilde{c} \notin \mathbb{R}^N_+ \), on the other hand, we still have \( \tilde{c} \) is sharper than the GK bound. If \( \tilde{c} \) is to randomly generate \( c \) in \( \mathbb{R}^N_+ \) and compute (20). In the examples, we generate 1000 values for \( c \) and show the largest obtained value for (20).

From Table I, one remarks that for Systems II, III and VIII we have \( \tilde{c} \in \mathbb{R}^N_+ \), so that the new bound (20) with \( c = \tilde{c} \) is sharper than the GK bound, as expected. Also, the new bound (20) can be further improved by randomly generating additional \( c \) values as shown in the table. Furthermore, the PG bound which uses sums of joint probabilities of three events, may be even poorer (e.g., see Systems I and VI) than the PG bound which uses sums of joint probabilities of two events, \( \tilde{c} \). It is also weaker than (20) in several cases (see Systems I-IV). Finally, our numerical bound (27) is always sharper than the other tested bounds, and coincides with the optimal bound (26) with exponential complexity in \( N \) since \( N < 7 \) holds for these examples.

**TABLE I**

<table>
<thead>
<tr>
<th>System</th>
<th>I</th>
<th>I1</th>
<th>I2</th>
<th>I3</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>( P(\bigcup_{i=1}^{N} A_i) )</td>
<td>0.7890</td>
<td>0.6740</td>
<td>0.7890</td>
<td>0.9687</td>
<td>0.3900</td>
<td>0.3252</td>
<td>0.5346</td>
<td>0.5854</td>
<td></td>
</tr>
<tr>
<td>KAT Bound [11]</td>
<td>0.7247</td>
<td>0.6227</td>
<td>0.7222</td>
<td>0.8909</td>
<td>0.3833</td>
<td>0.2769</td>
<td>0.4434</td>
<td>0.5142</td>
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</tr>
<tr>
<td>GK Bound [3], [4]</td>
<td>0.7601</td>
<td>0.6510</td>
<td>0.7508</td>
<td>0.9231</td>
<td>0.3813</td>
<td>0.2972</td>
<td>0.4750</td>
<td>0.5390</td>
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<tr>
<td>PG Bound [6]</td>
<td>0.7443</td>
<td>0.6434</td>
<td>0.7556</td>
<td>0.9148</td>
<td>0.3900</td>
<td>0.3240</td>
<td>0.5271</td>
<td>0.5726</td>
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<tr>
<td>Analytical Bound [2, Eq. (7)]</td>
<td>0.7247</td>
<td>0.6227</td>
<td>0.7222</td>
<td>0.8909</td>
<td>0.3900</td>
<td>0.3205</td>
<td>0.4562</td>
<td>0.5464</td>
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<tr>
<td>Numerical Bound [2, Eq. (5)]</td>
<td>0.7487</td>
<td>0.6598</td>
<td>0.7422</td>
<td>0.9044</td>
<td>0.3900</td>
<td>0.3252</td>
<td>0.5090</td>
<td>0.5531</td>
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</tr>
<tr>
<td>New Bound (20) with ( c = \tilde{c}^+ )</td>
<td>0.7638</td>
<td>0.6517</td>
<td>0.7512</td>
<td>0.9231</td>
<td>0.3900</td>
<td>0.2951</td>
<td>0.4905</td>
<td>0.5412</td>
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</tr>
<tr>
<td>New Bound (20) with random ( c )</td>
<td>0.7783</td>
<td>0.6633</td>
<td>0.7810</td>
<td>0.9501</td>
<td>0.3900</td>
<td>0.3203</td>
<td>0.4992</td>
<td>0.5606</td>
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<tr>
<td>New Numerical Bound (27)</td>
<td>0.7890</td>
<td>0.6740</td>
<td>0.7890</td>
<td>0.9687</td>
<td>0.3900</td>
<td>0.3027</td>
<td>0.5090</td>
<td>0.5673</td>
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</tr>
</tbody>
</table>

**REFERENCES**