Logistic Regression with more than two outcomes

- Ordinary logistic regression has a linear model for one response function
- Multinomial logit models for a response variable with c categories have c-1 response functions.
- Linear model for each one
- It’s like multivariate regression.
Model for three categories

\[
\begin{align*}
\ln \left( \frac{\pi_1}{\pi_3} \right) &= \beta_{0,1} + \beta_{1,1}x_1 + \ldots + \beta_{p-1,1}x_{p-1} \\
\ln \left( \frac{\pi_2}{\pi_3} \right) &= \beta_{0,2} + \beta_{1,2}x_1 + \ldots + \beta_{p-1,2}x_{p-1}
\end{align*}
\]

Need \textit{k-1 generalized logits} to represent a dependent variable with \textit{k} categories
Meaning of the regression coefficients

A positive regression coefficient for logit $j$ means that higher values of the independent variable are associated with greater chances of response category $j$, compared to the reference category.

\[
\ln \left( \frac{\pi_1}{\pi_3} \right) = \beta_{0,1} + \beta_{1,1} x_1 + \ldots + \beta_{p-1,1} x_{p-1}
\]

\[
\ln \left( \frac{\pi_2}{\pi_3} \right) = \beta_{0,2} + \beta_{1,2} x_1 + \ldots + \beta_{p-1,2} x_{p-1}
\]
Solve for the probabilities

\[
\ln \left( \frac{\pi_1}{\pi_3} \right) = L_1
\]

\[
\ln \left( \frac{\pi_2}{\pi_3} \right) = L_2
\]

So

\[
\frac{\pi_1}{\pi_3} = e^{L_1}
\]

\[
\frac{\pi_2}{\pi_3} = e^{L_2}
\]

\[
\pi_1 = \pi_3 e^{L_1}
\]

\[
\pi_2 = \pi_3 e^{L_2}
\]
Three linear equations in 3 unknowns

\[ \pi_1 = \pi_3 e^{L_1} \]

\[ \pi_2 = \pi_3 e^{L_2} \]

\[ \pi_1 + \pi_2 + \pi_3 = 1 \]
Solution

\[
\pi_1 = \frac{e^{L_1}}{1 + e^{L_1} + e^{L_2}}
\]

\[
\pi_2 = \frac{e^{L_2}}{1 + e^{L_1} + e^{L_2}}
\]

\[
\pi_k = \frac{1}{1 + e^{L_1} + e^{L_2}}
\]
In general, solve $k$ equations in $k$ unknowns

\[
\begin{align*}
\pi_1 &= \pi_k e^{L_1} \\
\vdots \\
\pi_{k-1} &= \pi_k e^{L_{k-1}} \\
\pi_1 + \cdots + \pi_k &= 1
\end{align*}
\]
General Solution

\[
\begin{align*}
\pi_1 &= \frac{e^{L_1}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\
\pi_2 &= \frac{e^{L_2}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\
\vdots \\
\pi_{k-1} &= \frac{e^{L_{k-1}}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\
\pi_k &= \frac{1}{1 + \sum_{j=1}^{k-1} e^{L_j}}
\end{align*}
\]
Using the solution, one can

• Calculate the probability of obtaining the observed data as a function of the regression coefficients: Get maximum likelihood estimates (\( \beta\)-hat values)
• From maximum likelihood estimates, get tests and confidence intervals
• Using \( \beta\)-hat values in \( L_j \), estimate probabilities of category membership for any set of \( x \) values.
R’s mlogit package

• Not part of the base installation
• You need to download it
• Can (should) do so from within R
Getting the mlogit package

- In Packages and Data, select Package Installer.
- Click on Get List.
- Maybe pick a mirror site.
- Select mlogit from a long list of packages.
- With Install Dependencies selected, click Install Selected.
- Once installation is finished, quit R.
- Start R again.
- Type library(mlogit), or in Packages and Data, select Package Manager and check mlogit.
Handle with Care

- The mlogit package is complicated and tricky to use compared to core R functions like lm and glm.
- I can shield you from most of it.
- But it requires a special kind of data frame.
- There’s a function for converting an ordinary data frame to one of the kinds mlogit can use.
- And the syntax of the model specification is unusual.
The complexity is justified

• Because the mlogit function can do a lot more than the multinomial logit model presented here.

• In addition to explanatory variables specific to the individual (like income), there can be explanatory variables specific to the categories of the response variable.

• Like if the response is what car the person buys, the prices of the cars can be an explanatory variable.
It gets even better

- There can even be alternative-specific explanatory variables that are different for different individuals, like the years of experience of the salesperson who was selling each type of car that day.
- And the model can accommodate several choices among the same set of alternatives by each individual. Like try the coffees three times.
It’s really impressive

• The models can seemingly allow the discrete outcomes to be determined by unobservable continuous variables – a kind of threshold idea.

• This was designed by econometricians; can you tell?

• They are interested in economic choices.

• We will be less ambitious, and focus on logistic regression for a multinomial response variable with 2 or more categories.

• This will allow us to avoid most of the extra complexity, but not all.
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