STA 257 – Fall 2002

Practice Problems 7
Recommended preparation for quiz to be held in tutorial on Wednesday, November 13

Sections from Schaeffer covered the week of November 4: 6.1, 6.2, 6.3, 4.5 \((N(\mu, \sigma^2)\) distribution), convolution

**Questions from the textbook (Schaeffer):**

1. From Section 6.2: 6.1, 6.3, 6.5, 6.7
2. From Section 6.4 (exercises covering material from 6.3): 6.9
3. From Section 4.6: 4.65

**Additional questions:**

4. Let \(X\) have an exponential distribution with mean \(1/\lambda\). Given that \(X = x\), let \(Y\) conditionally have the normal distribution with expected value \(\mu X\) and variance \(\sigma^2 X^2\). Find \(EY\) and \(VY\). [Ans: \(EY = \mu \lambda\), \(VY = (\mu^2 + 2\sigma^2)/\lambda^2\)]

5. The (standard) lognormal distribution is the distribution of a random variable \(Y\) for which \(X = \ln Y\) has a standard normal distribution. Find a density for \(Y\). [Ans: \(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}}, y > 0\)]

6. Let the cumulative distribution function of \(X\) be

\[
F_X(x) = \begin{cases} 
0 & \text{if } x < 0 \\
x^2/4 & \text{if } 0 \leq x < 2 \\
1 & \text{if } x \geq 2 
\end{cases}
\]

Let \(Y = X^2/4\). Find the distribution of \(Y\). [Ans: Uniform on \((0, 1)\)]

7. Suppose that \(f(y)\) is a density, and that we would like to generate observations of a random variable \(Y\) having this density using a computer. The computer gives us observations of a random variable \(X\) having the uniform distribution on \((0, 1)\).

(a) Show that if \(F(y)\) is the distribution function corresponding to the density \(f(y)\), and \(y = h(x)\) is the inverse of the function \(x = F(y)\), then the random variable \(Y = h(X)\) has the desired density.

(b) Let the desired density be \(f(y) = 2/y^3\) for \(y > 1\) (and 0 otherwise). Find the transformation \(Y = h(X)\) that produces a \(Y\) with this density.

[Ans: (b) \(Y = 1/\sqrt{1-X}\)]

8. Let the joint density of \(X\) and \(Y\) be

\[
f(x, y) = \begin{cases} 
6e^{-3x-2y} & \text{if } x > 0 \text{ and } y > 0 \\
0 & \text{otherwise}
\end{cases}
\]

Find a density for \(Z = X + Y\) by each of the following approaches:

(a) Using the cumulative distribution function approach.
(b) Noting that $X$ and $Y$ are independent random variables, finding the marginal densities of $X$ and $Y$ and using the convolution theorem.

[Ans: $6e^{-2z} - 6e^{-3z}$ for $z > 0$]

9. If $X \sim N(15, 30)$ and $Y \sim N(-12, 15)$ find the following:

(a) The distribution of $X + Y$.
(b) The distribution of $X - 2Y$.
(c) The probability that $2X + 3Y \leq 5$.
(d) The probability that $X > Y$.

[Ans: (a) $N(3, 45)$ (b) $N(39, 90)$ (c) .7549 (d) Essentially 1]

10. Let $X_1, X_2, \ldots, X_{30}$ be a sample from the $N(20, 25)$ distribution. Find the following:

(a) The probability that $\overline{X}_{30}$ is between 19 and 21.
(b) The probability that $S_{30} = X_1 + X_2 + \ldots + X_{30}$ is greater than 650.

[Ans: (a) .7286 (b) .0336]

11. Find the number $a$ so that the probability is 95% that a normally distributed random variable is within $a$ standard deviations of its expected value. [Ans: 1.96]

Material to be covered during the week of November 11 and relevant sections in Schaeffe: Gamma distribution (4.5) and Chi-square distribution, Multivariate change of variables theorem (not in textbook), other probability distributions which are useful in statistics.