1. (5 points) Suppose there are three cabinets labelled $A$, $B$, and $C$, each of which has two drawers. Each drawer contains one coin. In cabinet $A$ there are two gold coins, in cabinet $B$ there are two silver coins, and in cabinet $C$ there is one gold and one silver coin. A cabinet is chosen at random, one of the drawers is opened, and a silver coin is found. What is the probability that the other drawer in that cabinet contains a silver coin?

Let $A$, $B$, $C$ be the events that cabinets $A$, $B$, $C$ resp. were chosen.

Let $S$ be the event discovered coin is silver.

Want $P(B|S)$.

By Bayes’ Theorem:

$$P(B|S) = \frac{P(S|B)P(B)}{P(S|A)P(A) + P(S|B)P(B) + P(S|C)P(C)}$$

$$= \frac{\frac{1}{3}}{0 + \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}} = \frac{2}{3}$$
2. (10 points) Prove each of the following.

(a) For any two events $A$ and $B$, $P(A \cap B) \geq 1 - P(A^c) - P(B^c)$.

\[ \text{Know: } P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

\[ \Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B) \]

\[ = 1 - P(A^c) + 1 - P(B^c) - P(A \cup B) \]

\[ = 1 - P(A^c) - P(B^c) + 1 - P(A \cup B) \]

\[ \geq 1 - P(A^c) - P(B^c) \]

Since $1 - P(A \cup B) \geq 0$

Since $0 \leq P(A \cup B) \leq 1$

(b) If $P(B|A^c) = P(B|A)$, then $A$ and $B$ are independent.

If $P(B|A^c) = P(B|A)$

then $P(B \cap A^c) = \frac{P(B \cap A)}{P(A^c)} = \frac{P(B \cap A)}{P(A)}$

so $P(A)P(B \cap A^c) = P(B \cap A)(1 - P(A))$

$P(A)[P(B \cap A^c) + P(B \cap A)] = P(B \cap A)$

so $P(A)P(B) = P(B \cap A)$ (LT. P.)

So $A, B$ are indep.
3. (15 points) Let $X$ be a discrete random variable with probability mass function $P(X = n) = (1 - \theta)\theta^{n-1}$, $n = 1, 2, 3, \ldots$, $(0 < \theta < 1)$.

(a) Show $P(X > n) = \theta^n$.

$$P(X > n) = \sum_{k=n+1}^{\infty} (1 - \theta)\theta^{k-1} = (1 - \theta)\theta^n \sum_{j=0}^{\infty} \theta^j$$

$$= \frac{(1 - \theta)\theta^n}{1 - \theta} = \theta^n$$

(b) Show $P(X > m + n \mid X > m) = P(X > n)$.

$$P(X > m + n \mid X > m) = \frac{P(X > m + n, X > m)}{P(X > m)} = \frac{P(X > m + n)}{P(X > m)}$$

$$= \frac{\theta^{m+n}}{\theta^m} \quad \text{from (a)}$$

$$= \theta^n = P(X > n)$$

(c) Let $X_1$ and $X_2$ be independent discrete random variables with probability mass functions

$$P(X_1 = n) = (1 - \theta_1)\theta_1^{n-1}, \quad n = 1, 2, 3, \ldots \quad (0 < \theta_1 < 1)$$

$$P(X_2 = n) = (1 - \theta_2)\theta_2^{n-1}, \quad n = 1, 2, 3, \ldots \quad (0 < \theta_2 < 1)$$

Show $P(X_2 > X_1) = \frac{\theta_2(1 - \theta_1)}{1 - \theta_1\theta_2}$.

$$P(X_2 > X_1) = \sum_{n=1}^{\infty} P(X_2 > n \mid X_1 = n) \cdot P(X_1 = n)$$

$$= \sum_{n=1}^{\infty} \theta_2^n (1 - \theta_1)\theta_1^{n-1}$$

$$= \theta_2 (1 - \theta_1) \sum_{k=0}^{\infty} (\theta_1 \theta_2)^k$$

$$= \theta_2 (1 - \theta_1) \frac{1}{1 - \theta_1 \theta_2}$$

Total Pages = (10)
4. (4 points) A random variable $X'$ is said to be obtained from the random variable $X$ by “truncation at the point $a$” if $X'$ is defined by

$$X'(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) \leq a \\ a & \text{if } X(\omega) > a \end{cases}$$

Express the distribution function of $X'$ in terms of the distribution function of $X$.

$$F_{X'}(x) = P(X' \leq x)$$

$$= \begin{cases} F_X(x) & , X(\omega) \leq a \\ 1 & , X(\omega) > a \end{cases}$$

5. (7 points) $X$ and $Y$ are jointly distributed discrete random variables with joint mass function given in the table:

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>0</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>1</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>.1</td>
<td>.05</td>
<td>?</td>
</tr>
</tbody>
</table>

Using the information that $P(Y = 2|X = 0) = \frac{1}{4}$ and that $X$ and $Y$ are independent, fill in the missing information in the table.

$$P(Y = 2) = 0.25 \quad \text{(since indep.)}$$

So

$$P(6, 2) = 0.1$$

Since indep.,

$$P(6, 2) = p_X(6) p_Y(2)$$

$$0.1 = p_X(6) \cdot 0.25$$

So

$$p_X(6) = 0.4$$

So

$$p(6, 1) = 0.3$$

Similarly,

$$P(3, 2) = p_X(3) p_Y(2)$$

$$0.05 = p_X(3) \cdot 0.25$$

$$p_X(3) = 0.2$$

So

$$P(3, 1) = 0.15$$

By subtraction

$$P(0, 1) = 0.3$$

Total Pages = (10)
6. (17 points) Suppose \( X \) and \( Y \) are uniformly distributed over the triangle with vertices \((-1, 0)\), \((0, 1)\), and \((1, 0)\).

(a) What is the joint density function of \( X \) and \( Y \)?

\[
\begin{align*}
\text{density: } f_{X,Y}(x, y) &= \begin{cases} 
1 & 0 \leq y \leq 1, \quad y + |x| \leq 1, \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

(b) Find \( P(X \leq \frac{3}{4}, Y \leq \frac{3}{4}) \).

\[
= 1 - \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{4} \right) - \frac{1}{2} \left( \frac{1}{4} \right) \left( \frac{1}{4} \right)
\]

\[
= 1 - \frac{1}{16} - \frac{1}{32} = \frac{29}{32}
\]

(c) Find \( E(XY) \).

\[
E(XY) = \int_{-1}^{1} \int_{0}^{1} xy \, dy \, dx + \int_{-1}^{1} \int_{0}^{1-x} xy \, dy \, dx
\]

\[
= \int_{-1}^{1} \frac{x y^2}{2} \bigg|_{y=0}^{y=1-x} \, dx + \int_{-1}^{1} \frac{x y^2}{2} \bigg|_{y=0}^{y=x-1} \, dx
\]

\[
= \int_{-1}^{1} \frac{x (1-x)^2}{2} \, dx + \int_{-1}^{1} \frac{x (1-x^2)}{2} \, dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} \left( x + 2x^2 + x^3 \right) \, dx + \frac{1}{2} \int_{0}^{1} \left( -x - 2x^2 + x^3 \right) \, dx
\]

\[
= \frac{1}{2} \left\{ \left[ \frac{x^2}{2} + \frac{2}{3} x^3 + \frac{x^4}{4} \right]_{-1}^{1} \right\} + \frac{1}{2} \left\{ \left[ \frac{x^2}{2} - \frac{2}{3} x^3 + \frac{x^4}{4} \right]_{0}^{1} \right\}
\]

\[
= \frac{1}{2} \left\{ -\frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right\}
\]

\[
= 0
\]
7. (5 points) Suppose \( X \) and \( Y \) are discrete random variables with joint probability mass function \( p(x, y) \). Prove \( E(aX + bY) = aE(X) + bE(Y) \), where \( a, b \in \mathbb{R} \).

\[
E(aX + bY) = \sum_x \sum_y (aX + bY) p(x, y) \\
= a \sum_x \sum_y x p(x, y) + b \sum_y \sum_x y p(x, y) \\
= a \sum_x xp(x) + b \sum_y y p_y(y) \\
= a E(X) + b E(Y)
\]

8. (5 points) If \( X \sim \text{Unif}(0, 1) \) find the density of \( Y = -2 \log(X) \).

Let \( h(x) = -2 \log x \Rightarrow \)

Since \( h(x) \) is monotone, can apply the thm.

\[
h^{-1}(y) = e^{-\frac{y}{2}} \\
\frac{d}{dy} h^{-1}(y) = -\frac{1}{2} e^{-\frac{y}{2}}
\]

\[
f_Y(y) = \begin{cases} 
\frac{1}{2} e^{-\frac{y}{2}}, & y > 0 \\
0, & \text{otherwise}
\end{cases}
\]

Total Pages = (10)
9. (15 points) Suppose $X$ and $Y$ are independent, identically distributed exponential random variables with parameter $\lambda$.

(a) Find $P(X \geq Y \geq 2)$.

$$f_{x,y}(x,y) = x^2 e^{-x} e^{-y}, \quad x, y > 0$$

$$P(X \geq Y \geq 2) = \int_2^\infty \int_2^x 2xe^{-x} e^{-y} dy \, dx$$

$$= \int_2^\infty -2e^{-x} e^{-y} \bigg|_{y=2}^x \, dx$$

$$= \int_2^\infty -2e^{-x} e^{-2} + \frac{1}{2} e^{-2x} \bigg|_{x=2}^{x=\infty} = \frac{1}{2} e^{-4}$$

(b) Find the joint density function of $U = \frac{X}{Y}$ and $V = X + Y$. Are $U$ and $V$ independent?

$$x = uy, \quad y = v - \frac{ux}{1+u}$$

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial y}{\partial u} = \frac{-v}{(1+u)^2}$$

$$JACOBIAN \left| \begin{array}{cc}
\frac{v}{(1+u)^2} & \frac{1}{1+u} \\
-\frac{v}{(1+u)^2} & \frac{1}{1+u}
\end{array} \right| = \frac{v}{(1+u)^3} + \frac{uv}{(1+u)^3} = \frac{v}{(1+u)^2}$$

$$f_{u,v}(u,v) = \begin{cases} x^2 e^{-x} \frac{v}{(1+u)^2}, & u,v > 0 \\
0, & \text{otherwise}
\end{cases}$$
10. (3 points) Suppose $T$ has a $t$ distribution with $n$ degrees of freedom. What is the distribution of $T^2$? State the value of any parameters.

\[ T^2 \sim F(1, n). \]

11. (8 points) Use probability generating functions to find the distribution of $Y = X_1 + X_2$ where $X_1$ and $X_2$ are Poisson random variables with parameters $\lambda_1$ and $\lambda_2$, respectively. (The probability mass function for the Poisson distribution is $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \ldots$)

Probability generating function for $X_1$:

\[ \pi_X(t) = \sum_{k=0}^{\infty} \frac{\lambda_1^k e^{-\lambda_1}}{k!} t^k = e^{-\lambda_1} e^{\lambda_1 t} = e^{-\lambda_1 (1-t)} \]

Probability generating function for $X_1 + X_2$:

\[ \pi_{X_1+X_2}(t) = \pi_{X_1}(t) \pi_{X_2}(t) \]

\[ = e^{-\lambda_1 (1-t)} e^{-\lambda_2 (1-t)} \]

\[ = e^{-\lambda_1 (1-t) - \lambda_2 (1-t)} = e^{-\lambda_1 + \lambda_2 (1-t)} \]

which is the p.g.f. for a Poisson r.v.

with parameter $\lambda_1 + \lambda_2$

So $Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.
12. (6 points) For each of the following functions, state whether or not it is a moment generating function. If not, explain why not. If so, find the underlying distribution.

(a) \( m(t) = \frac{e^t}{4 - e^t} \)

\[ m(0) = \frac{1}{3} \neq 1 \]

so not a mgf.

(b) \( m(t) = \frac{3e^{4t} + e^{-2t}}{4} \) (\( m(0) = 1 \))

Since \( m(t) = E(e^{tx}) \)

this is the mgf for a discrete r.v. \( X \) whose

mass function is \( p(4) = \frac{3}{4}, p(-2) = \frac{1}{4} \).

13. (5 points) Let \( m(t) \) be the moment generating function of the random variable \( X \) and define \( \kappa(t) = \log m(t) \). Show that

\[ \frac{d}{dt} \kappa(t) \bigg|_{t=0} = E(X) \]

and

\[ \frac{d^2}{dt^2} \kappa(t) \bigg|_{t=0} = \text{Var}(X). \]

\[ \frac{d}{dt} K(t) \bigg|_{t=0} = \frac{1}{m(t)} m'(t) \bigg|_{t=0} = E(X) \quad \text{since} \quad m'(0) = E(X) \quad m(0) = 1 \]

\[ \frac{d^2}{dt^2} K(t) \bigg|_{t=0} = \frac{m(t)m''(t) - [m'(t)]^2}{[m(t)]^2} \bigg|_{t=0} \quad \text{Total Pages} = (10) \]

\[ = E(X^2) - \frac{[E(X)]^2}{E(X)} \quad \text{since} \quad m''(0) = E(X^2) \]

\[ = \text{Var}(X) \]
14. (5 points) Let $X$ be a non-negative random variable such that its moment generating function, $m(t)$, is finite for all $t$. Prove $P(X \geq a) \leq e^{-ta}m(t)$ for $t \geq 0$, and $a$ a positive constant.

$$P(X \geq a) = P(tX \geq ta) \quad \text{for } t > 0$$

$$= P(e^{tX} \geq e^{ta}) \quad \text{since } e^x \text{ is monotone incr.}$$

$$\leq \frac{E(e^{tX})}{e^{ta}} \quad \text{by Chebychev's Inequality}$$

$$= e^{-ta}m(t)$$

15. (5 points) A fair die is rolled 12000 times. Let $S$ be the total number of sixes. Use the Central Limit Theorem to find $P(1900 < S < 2200)$ in terms of $\Phi$, the standard normal distribution function.

$$E(S) = \frac{12000}{6} = 2000$$

$$Var(S) = 12000 \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{5000}{3}$$

by CLT, $S \sim N(2000, \frac{5000}{3})$

$$P(1900 < S < 2200)$$

$$= P(S < 2200) - P(S < 1900)$$

$$\approx \Phi\left(\frac{2200 - 2000}{\sqrt{\frac{5000}{3}}}\right) - \Phi\left(\frac{1900 - 2000}{\sqrt{\frac{5000}{3}}}\right)$$

Total Pages = (10)
Total Points = (115)