with parameter
- \( a \) is Greek symbol for parameter(s)
  - Normal distribution: \( \mu, \sigma^2 \) or \( \sigma \)
  - Poisson: \( \lambda \)
  - Binomial: \( n, p \)
    - Usually considered known
  - Usual situation: don't know parameters

Goal of statistics: (from a theoretical point-of-view)
- Estimate parameter
- Measure errors of these estimates
- Test whether sample of data gives evidence that parameters have (or are not) a certain value

Two theoretical results from STA 247

Weak Law of Large Numbers (LLN or WLLN)
- Let \( X_1, X_2, X_3, \ldots \) sequence of i.i.d. r.v.'s with \( E(X_i) = \mu \), \( \text{Var}(X_i) = \sigma^2 \)
- Let \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \)

\[
\lim_{n \to \infty} P \left( \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq x \right) = \Phi(x)
\]

\( \bar{X}_n \) standardized where \( \Phi(x) \) is the standard normal cdf

\[
\text{Var}(\bar{X}_n) = \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right)
= \frac{1}{n^2} \text{Var}(\sum_{i=1}^{n} X_i)
= \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(X_i)
= \frac{1}{n^2} (n \sigma^2)
= \sigma^2 / n
\]

Then for any \( \varepsilon > 0 \)
- \( P(|\bar{X}_n - \mu| > \varepsilon) \to 0 \) as \( n \to \infty \)

Central Limit Theorem (CLT)
- Let \( X_1, X_2, X_3, \ldots \) be a sequence of i.i.d. r.v.'s having \( E(X_i) = \mu \), and \( \text{Var}(X_i) = \sigma^2 \)

\[
\lim_{n \to \infty} P \left( \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq x \right) = \Phi(x)
\]

\( \bar{X}_n \) standardized where \( \Phi(x) \) is the standard normal cdf

\[
\text{Var}(\bar{X}_n) = \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right)
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= \frac{1}{n^2} (n \sigma^2)
= \sigma^2 / n
\]

Estimator:
- An estimator \( \hat{\theta} \) or \( \hat{\theta}_n \) of an unknown parameter \( \theta \) is a function statistic to estimate \( \theta \)
- estimator = function of \( X_1, \ldots, X_n \)
- estimate = function of \( \bar{X}_n, X_1, \ldots, X_n \)

What makes a good estimator?
- Unbiased
- Consistent
- Small variance - variance decreases as sample size increases
- (Know its sampling distribution)

Unbiased
- \( E(\hat{\theta}) = \theta \)
- May not always exist
- May be ridiculous
- \( \hat{\theta} \) is the mean of Poisson distribution
- Want to estimate \( e^{\lambda} \)

\[
\hat{\theta} = \frac{1}{n} X_{\text{even}}
\]

\[
E(\hat{\theta}) = e^\lambda
\]

\[
\hat{\theta} = \frac{1}{n} X_{\text{even}}
\]

\( X_{\text{odd}} \)
- But $e^{-x}$ is between 0 and 1 and $8(x)$ never is.

- $\theta$ may be unbiased for $\theta$ but this does not necessarily mean that $g(\theta)$ is unbiased for $g(\theta)$.

**Example** 
\[ \begin{align*}
(X_1, \ldots, X_n) \overset{\text{i.i.d.}}{\sim} & \quad \text{and } \text{Var}(X_i) = \sigma^2 \\
E(X_i) &= \mu \\
\end{align*} \]

**Mean:** $\mu$

**Usual estimator:**
\[ \hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n} \]

*Unbiased?* $E(\hat{\mu}) = \mu$?

**Proof:**
\[ \begin{align*}
E(\hat{\mu}) &= E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n} E\left(\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n} \left( \sum_{i=1}^{n} E(X_i) \right) \\
&= \frac{1}{n} \left( \sum_{i=1}^{n} \mu \right) \\
&= \mu \\
\end{align*} \]

So $\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$ is unbiased for $\mu$.

**Variance:**
\[ \sigma^2 = E(X_i - \mu)^2 \]

**Estimator:**
\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \]

*Unbiased?*

\[ \begin{align*}
E(\hat{\sigma}^2) &= E\left(\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2\right) \\
&= \frac{1}{n} E\left(\sum_{i=1}^{n} (X_i - \bar{X})^2\right) \\
&= \frac{1}{n} \sum_{i=1}^{n} E(X_i - \bar{X})^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} \text{Var}(X_i) \\
&= \frac{1}{n} (n \sigma^2) \\
&= \sigma^2 \\
\end{align*} \]

$\hat{\sigma}^2$ is unbiased for $\sigma^2$. 

**Variance:**
\[ \sigma^2 = E(X_i - \bar{X})^2 \]

\[ \begin{align*}
\hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \\
&= \frac{\sum_{i=1}^{n} (X_i - \bar{X})}{n} \\
&= \frac{\sum_{i=1}^{n} X_i - n \bar{X}}{n} \\
&= 0 \\
\end{align*} \]