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Non-stationary structural model with time-varying demand elasticities

Kun Ho Kim [a], Zhou Zhou [b], Wei Biao Wu [b,*]

[a] *Department of Economics, University of Chicago, 1126 E. 59th St., Chicago, IL 60637, USA*
[b] *Department of Statistics, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA*

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ABSTRACT

The paper considers local linear regression of a time series model with non-stationary regressors and errors. Asymptotic property of the local linear estimator is derived under a new dependence measure of non-stationary time series. We apply the local linear regression method to estimate the "time-varying" coefficients of an economic-causal model for the industrial sector of the U.S. economy. Nonparametric bootstrap test on the time-varying coefficients strongly suggests that the price/income elasticities of the U.S. durable goods demand are time-varying.

1. Introduction

Time-varying models have been increasingly popular in econometrics (Cooley and Prescott, 1973, 1976; Chen and Tsay, 1993; Hastie and Tibshirani, 1993; Reyes, 1999; Orbe et al., 2005, 2006). In general, these models can not only capture the dynamic patterns in econometric modeling effectively but also explain many interesting physical and social phenomena that models with time-invariant parameters cannot (Fan and Zhang, 2008). In this paper, we perform estimation and inference of time-varying coefficient models using local linear regression (Fan and Gijbels, 1996). Traditionally, a linear model with time-varying coefficients assumes the stationarity of regressors and errors (Cai, 2007). However, in many econometric models, regressors and errors are often non-stationary. Here we relax the stationarity assumption.

Specifically, we allow the regressors and the errors of our model to be locally stationary, which is a mild form of non-stationarity. For backgrounds on locally stationary processes, see Priestley (1965), Dahlhaus (1997), Mallat et al. (1998), Ombao et al. (2005) and Zhou and Wu (2009). See also Section 2.1 for more discussion. Given the non-stationary nature of model regressors and errors, we derive asymptotic properties of local linear estimates (see Appendix A). As an application of this non-stationary framework, we propose an economic-causal model with time-varying parameters. The reduced-form equation of the developed model has non-stationary regressors and errors, which makes our modified framework on time-varying coefficient model highly applicable.

Time-varying parametrization of economic models has been used extensively in the literature. See Priestley (1980, 1988) for state-dependent models, and Lundbergh et al. (2003) for time-varying smooth transition autoregressive models. As argued in Cooley and Prescott (1976), Teräsvirta and Anderson (1992), McCulloch and Tsay (1994), Teräsvirta (1994), Stock and Watson (1996) and particularly in Phillips (2001), structural change is one of the key aspects in economic model building. Models with time-invariant parameters are inherently vulnerable to possible structural changes in the economy. Consequently, the estimation results based on such models could be misleading if the model parameters indeed change

* Corresponding author.
E-mail address: wbwu@galton.uchicago.edu (W.B. Wu).

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over time. Therefore, modeling time variation in structural parameters can help avoid such a problem and allows us to incorporate structural changes effectively.

In this work, we construct a structural model with time-varying demand elasticity. Elasticity of demand measures the change in demand due to a unit increase in either market price or income. Although the parameter has been treated time-invariant in traditional models, there has been a growing number of works claiming that demand elasticity indeed changes over time (Hackl and Westlund, 1995, 1996; Orbe et al., 2006). We model time-varying elasticity in our framework and develop an economic-causal model with time-varying coefficients. Then, the local linear regression and the bootstrap test are employed to estimate and to test the model coefficient functions. The test result strongly suggests that the time-invariant-parameter framework is not suitable for modelling industrial sectors of the U.S. economy.

The contribution of our paper is twofold. First, we allow the predictors and the error term of model (1) to be non-stationary. The asymptotic consistency of local linear estimators is established under the non-stationary assumption of model predictors and errors. Second, we present empirical evidence of time-varying demand elasticities for the durable manufacturing section of the U.S. economy. While traditional models employ time-invariant coefficients, we argue that the parameters indeed change over time.

The organization of the paper is as follows. Section 2 introduces the nonparametric estimation methods. Local stationarity and local linear regression will be discussed in details. Section 3 derives an economic-causal model with time-varying demand elasticities. Section 4 introduces the bootstrap specification test and explains the procedures of testing. Section 5 describes the data used in this paper. The estimation and the test result along with economic interpretations will be provided in Section 6. Section 7 concludes the paper. The asymptotic properties of the local linear estimator will be presented in Appendix A.

2. Time-varying coefficient (TVC) models

Consider the following model with time-varying coefficient $\boldsymbol{\beta}_j$, where $\boldsymbol{\beta}_j$ varies in time j:

$$y_j = \mathbf{x}_j^\top \boldsymbol{\beta}_j + \varepsilon_j, \quad j = 1,2,\ldots,n. \tag{1}$$

Here (y_j) is the observed response series, (ε_j) is the error process satisfying $\mathbb{E}(\varepsilon_j|\mathbf{x}_j)=0$ and $\mathbf{x}_j=(x_{j1},x_{j2},\ldots,x_{jp})^\top$ are the explanatory variables and we set $x_{j1}\equiv 1$. We are interested in estimation and inference of the regression coefficients $\boldsymbol{\beta}_j$ as a function of time j. As discussed earlier, we shall treat $\boldsymbol{\beta}_j$ as a smooth function of time and hence adopt the following formulation (Robinson, 1989):

$$\boldsymbol{\beta}_j = \boldsymbol{\beta}(j/n), \quad j=1,2,\ldots,n, \tag{2}$$

where $\boldsymbol{\beta}(\cdot) := (\beta_1(\cdot),\ldots,\beta_p(\cdot))^\top : [0,1] \to \mathbb{R}^p$ is assumed to be continuously differentiable. As argued in Robinson (1989), making $\boldsymbol{\beta}_j$ depend on the sample size n in the form of (2) is necessary to provide asymptotic justification for any nonparametric smoothing estimators so that one can have consistent estimates. Instead of making any specific parametric assumption on $\boldsymbol{\beta}(\cdot)$, we allow $\boldsymbol{\beta}(\cdot)$ to be a smooth function on [0,1] and let data determine its form. By letting $\boldsymbol{\beta}(\cdot)$ change slowly with respect to time, we can separate it from the stochastic noise of model (1) and hence estimation and inference of $\boldsymbol{\beta}(\cdot)$ can be theoretically guaranteed. Additionally, since $\boldsymbol{\beta}(\cdot)$ is continuous in time, model (1) can capture slowly changing dynamics and the relationship between $\mathbf{x}_j$ and y_j changes smoothly over time; see also Robinson (1989), Fan and Yao (2005) and Cai (2007). In the sequel we call (1) the "time-varying coefficient model".

2.1. Locally stationary time series models

Here we shall adopt a general class of non-stationary processes for predictors and errors. To this end, first note that since the major purpose of (1) is to examine the extent to which the relation between the response and predictors varies over time, it is often necessary to assume that the characteristics of the predictors and errors also vary with respect to time. In other words, we shall model the latter quantities as "non-stationary" time series. On the other hand, in economic applications, it is often reasonable to assume that the change of characteristics of variables is smooth with respect to time. Therefore if one observes an economic variable in a relatively short time span, this time series will be approximately stationary. For literature review on local stationarity, see Priestley (1965), Dahlhaus (1997), Mallat et al. (1998) and Ombao et al. (2005). Note, however, that most of these results concern time-varying properties for the first two moments such as time-varying spectra. Properties beyond the second moment will be lost in the spectra representations and hence it may not be useful in handling nonlinear features.

Following Draghicescu et al. (2009), we adopt the following general class of locally stationary time series models for the predictors and the errors:

$$\mathbf{x}_j = \mathbf{G}(j/n,\mathcal{F}_j) \quad \text{and} \quad \varepsilon_j = H(j/n,\mathcal{G}_j), \quad j=1,2,\ldots,n, \tag{3}$$

where $\mathcal{F}_j=(\ldots,\epsilon_{j-1},\epsilon_j)$ and $\mathcal{G}_j=(\ldots,\zeta_{j-1},\zeta_j)$ with $(\epsilon_j)_{j\in\mathbb{Z}}$ and $(\zeta_j)_{j\in\mathbb{Z}}$ being two sets of independent and identically distributed (iid) random elements, $\mathbf{G} := (G_1,G_2,\ldots,G_p)^\top : [0,1]\times\mathbb{R}^\infty \to \mathbb{R}^p$ and $H:[0,1]\times\mathbb{R}^\infty\to\mathbb{R}$ are measurable functions such that $\mathbf{G}(t,\mathcal{F}_j)$ and $H(t,\mathcal{G}_j)$ are well-defined for each $t\in[0,1]$.

If $\mathbf{G}(t,\mathcal{F}_j)$ and $H(t,\mathcal{G}_j)$ do not depend on t, then $\mathbf{x}_j$ and ε_j are stationary processes of a very general framework, and they represent many time series models used in practice. See Tong (1990) and Wu (2005) for more details. By allowing $\mathbf{G}$ and H varying continuously with t, we can have local stationarity of $(\mathbf{x}_j)$ and (ε_j). Formulation (3) can be viewed as physical systems with $\mathcal{F}_j$ and $\mathcal{G}_j$ being the inputs and $\mathbf{x}_j$, ε_j being the outputs, respectively, and $\mathbf{G}$ and H being the transforms or filters that represent the underlying physical mechanisms. The above formulation naturally extends most of the linear and nonlinear stationary time series models into the locally stationary regime and includes many existing locally stationary time series models as special cases (Zhou and Wu, 2009).

2.2. Local linear regression

Among many nonparametric methods, local polynomial regression (Fan and Gijbels, 1996) is frequently used due to its simple form, ease of computation and analytical tractability. Robinson (1989, 1991) and Orbe et al. (2005) used local constant estimates of $\boldsymbol{\beta}(\cdot)$. However, it is well-known that local linear estimator can suppress the boundary problem and achieve nearly optimal statistical efficiency (Fan and Gijbels, 1996) compared to its local constant counterpart. Thus, we shall use local linear estimate of $\boldsymbol{\beta}(\cdot)$. Let

$$(\hat{\boldsymbol{\beta}}_{b_n}(t),\hat{\boldsymbol{\beta}}'_{b_n}(t)) = \underset{(\eta_0,\eta_1)}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \mathbf{x}_i^\top \eta_0 - \mathbf{x}_i^\top \eta_1 (t_i - t))^2 K_{b_n}(t_i - t), \tag{4}$$

where K is the kernel function, $K_c(\cdot) = K(\cdot/c)$ for $c>0$, and b_n is the bandwidth. Here we let K be a symmetric probability density function with support $[-1,1]$. A popular choice is the Epanechnikov kernel $K(x)=3\max(1-x^2,0)/4$. Bandwidth b_n can be viewed as the size of the neighborhood on which $\hat{\boldsymbol{\beta}}_{b_n}(t)$ is estimated. We shall omit the subscript b_n of $\hat{\boldsymbol{\beta}}(t)$ in the sequel if no confusion will be caused. $\hat{\boldsymbol{\beta}}(\cdot)$ is called the "local linear estimate" of $\boldsymbol{\beta}(\cdot)$. As a by-product, (4) also gives an estimate of the derivative $\boldsymbol{\beta}'(\cdot)$. Since (4) is essentially a weighted least squares estimate, we have that

$$\hat{\boldsymbol{\eta}}(t) = \begin{pmatrix} \mathbf{S}_{n,0}(t) & \mathbf{S}_{n,1}^\top(t) \\ \mathbf{S}_{n,1}(t) & \mathbf{S}_{n,2}(t) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{R}_{n,0}(t) \\ \mathbf{R}_{n,1}(t) \end{pmatrix} \coloneqq \mathbf{S}_n^{-1}(t)\mathbf{R}_n(t), \tag{5}$$

where $\hat{\boldsymbol{\eta}}(t) = (\hat{\boldsymbol{\beta}}_{b_n}^\top(t), b_n(\hat{\boldsymbol{\beta}}'_{b_n}(t))^\top)^\top$, $\mathbf{S}_{n,i}(t) = (nb_n)^{-1}\sum_{j=1}^n \mathbf{x}_j\mathbf{x}_j^\top[(t_j-t)/b_n]^i K_{b_n}(t_j-t)$ and $\mathbf{R}_{n,i}(t) = (nb_n)^{-1}\sum_{j=1}^n \mathbf{x}_j y_j[(t_j-t)/b_n]^i K_{b_n}(t_j-t)$ for i=0,1,2. Asymptotic properties of the local linear estimator will be derived in Appendix A.

3. Structural model with TVCs

In this section, we derive a structural economic model with time-varying parameters based on a partial equilibrium framework for an individual sector of economy. As opposed to the popular non-structural model framework (Cooley and Prescott, 1973, 1976; Gallant and Fuller, 1973; Chen and Tsay, 1993), we here study the structural counterpart. The resulting equations possess both "non-stationary" predictors/errors and "time-varying" coefficients, which makes the previously discussed analysis relevant here.

First, we consider the industry-specific Cobb–Douglas production function $Q_i = A_i(t)L_i^{\alpha_i}K_i^{\beta_i}$ with fixed returns-to-scale parameters α_i and β_i such that $\alpha_i>0$, $\beta_i>0$ and $\alpha_i+\beta_i<1$ (Cobb and Douglas, 1928). Here $A_i(t)$ represents the time-varying level of production technology for sector i. Given labor L_i and capital K_i inputs, wage w and rent r, and product price P_i, we define sector-specific profit function as $\pi_i = P_iQ_i - wL_i - rK_i$ for sector i. In this sector-specific model, wage (w) and rent (r) are regarded "exogenous" and they do not vary across sector. Then, the producer's optimization problem is to maximize profit π_i under production function Q_i by choosing optimal K_i and L_i. We substitute the optimal K_i and L_i back into the production function and obtain the following sales supply for industry i:

$$S_i = N_iP_iQ_i = N_iP_i[A_1^i(t)P_i^{(\alpha_i+\beta_i)/(1-\alpha_i-\beta_i)} w^{-\alpha_i/(1-\alpha_i-\beta_i)} r^{-\beta_i/(1-\alpha_i-\beta_i)}], \tag{6}$$

where $A_1^i(t) = \alpha_i^{\alpha_i/(1-\alpha_i-\beta_i)}\beta_i^{\beta_i/(1-\alpha_i-\beta_i)}[A_i(t)]^{1/(1-\alpha_i-\beta_i)}$ and N_i is the number of firms in industry i. See Zellner and Israilevich (2005) and Kim (2010) for more details on how to derive (6). We take log and differentiate both sides of Eq. (6) in time:

$$\frac{\dot{S}_i}{S_i} = \frac{\dot{N}_i}{N_i} + \frac{1}{1-\alpha_i-\beta_i}\frac{\dot{A}_i(t)}{A_i(t)} - \frac{\alpha_i}{1-\alpha_i-\beta_i}\frac{\dot{w}}{w} - \frac{\beta_i}{1-\alpha_i-\beta_i}\frac{\dot{r}}{r} + \frac{1}{1-\alpha_i-\beta_i}\frac{\dot{P}_i}{P_i}. \tag{7}$$

On the demand side, we employ the following sales demand function with "time-varying elasticities" for industry i:

$$S_i = P_iHQ_i = P_iH[B_iP_i^{\eta_1(t)}S^{\eta_2(t)}], \tag{8}$$

where H is the number of households and S is the household income. Here B_i is a fixed parameter for the demand of sector i. Note that $\eta_1(t)$ is the price elasticity of demand (PED) for industry i and that $\eta_2(t)$ is the income elasticity of demand (IED). Zellner and Israilevich (2005) and Kim (2010) consider the similar demand equation with fixed elasticities. The crucial innovation here is that both $\eta_1(t)$ and $\eta_2(t)$ smoothly "evolve over time" in our model. Hackl and Westlund (1995, 1996) consider time-varying demand elasticities for the European telecommunication market and report the empirical evidence of time-variation in them. Orbe et al. (2006) introduce time-varying demand elasticity into their log-linear demand for meat. We make a similar assumption on the demand elasticities of our model. We take log and differentiate both sides of demand equation (8) in time:

$$\frac{\dot{S}_i}{S_i} = (1+\eta_1(t))\frac{\dot{P}_i}{P_i} + \dot{\eta}_1(t)\log P_i + \frac{\dot{H}}{H} + \eta_2(t)\frac{\dot{S}}{S} + \dot{\eta}_2(t)\log S. \tag{9}$$

The entry–exit equation has been often omitted in traditional economic models. It could be reasonable to assume that the number of firms in economy is constant in short run, but probably not for intermediate or long run analysis. In particular, Veloce and Zellner (1985) formulate a model for the Canadian furniture industry to illustrate the importance of including the entry–exit relation in analyzing the industry's behavior. Kim (2010) employs a disaggregate model with entry–exit equation to forecast the U.S. real GDP growth rate, and reports a significant gain in forecasting the variable with his model. Similarly, we address the issue by including the entry–exit relations in our model here. Let $F_i(t)$ be the "time-varying" equilibrium profit for the firms in industry i, and k_1 and k_2 be positive constants. As defined earlier, N_i is the number of firms in industry i. Then, the industry entry–exit equation is

$$\frac{\dot{N}_i}{N_i} = k_1(k_2 S_i - F_i(t)), \tag{10}$$

where k_2S_i is industry profit. The idea of (10) is that the number of firms in industry i will increase when $k_2S_i > F_i(t)$ and it will decrease if $k_2S_i < F_i(t)$. Similar equations are given in Zellner and Israilevich (2005) and Kim (2010). Here $F_i(t)$ is treated as time-varying.

Given (7), (9) and (10), we can solve for the growth rates of the three endogenous variables: S_i, P_i, and N_i. Orbe et al. (2006) directly estimate their simple demand equation for meat, leaving the problem of "endogeneity" unanswered in their work. In order to handle such a problem, we instead use the following "reduced-form" equations based on the supply, the demand and the entry–exit equation. In particular, the reduced-form equation for the growth rate of industry output S_i in continuous time is the following:

$$\begin{aligned}\frac{\dot{S}_i}{S_i} =& \left[\frac{k_1(1-\alpha_i-\beta_i)(1+\eta_1(t))}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}F_i(t) - \frac{1+\eta_1(t)}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\frac{\dot{A}_i(t)}{A_i(t)}\right]\\ &-\frac{k_1k_2(1-\alpha_i-\beta_i)(1+\eta_1(t))}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}S_i + \frac{\eta_2(t)}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\frac{\dot{S}}{S} + \frac{\alpha_i(1+\eta_1(t))}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\frac{\dot{w}}{w}\\ &+\frac{\beta_i(1+\eta_1(t))}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\frac{\dot{r}}{r} + \frac{1}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\frac{\dot{H}}{H}\\ &+\frac{\dot{\eta}_2(t)}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\log S + \frac{\dot{\eta}_1(t)}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\log P_i.\end{aligned} \tag{11}$$

We approximate $\dot{S}_i/S_i$ by $\log(S^i_{t+1}/S^i_t) = \dot{S}_i/S_i + \nu_{t+1}$, where $\{\nu_{t+1}\}$ are mean 0 random errors. Given discrete (quarterly) data for the output variable S_i (see Section 5), the log-linearization here gives us a reasonable approximation of the growth rate $\dot{S}_i/S_i$ in continuous time. Moreover, this approximation is widely used for modelling economic growth in discrete time. See Tiao and Tsay (1994), Zellner and Israilevich (2005) and Kim (2010) for more on this approximation. Similarly, the other growth rates in (11) can be approximated by the log-differences. The assumption of recursive structure (for example, AR(1)) on ν_{t+1} and on the other random errors from the log-difference approximation ensures that the resulting random error ε^i_{t+1} in (12) has the structure of model error term in (1). The continuous-time equation (11) then leads to the following equation in discrete time:

$$\begin{aligned}\log\left(\frac{S^i_{t+1}}{S^i_t}\right) =& \left[\frac{k_1(1-\alpha_i-\beta_i)(1+\eta_1(t))}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}F_i(t) - \frac{1+\eta_1(t)}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\frac{\dot{A}_i(t)}{A_i(t)}\right]\\ &-\frac{k_1k_2(1-\alpha_i-\beta_i)(1+\eta_1(t))}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}S^i_t + \frac{\eta_2(t)}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\log\left(\frac{S_{t+1}}{S_t}\right)\\ &+\frac{\alpha_i(1+\eta_1(t))}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\log\left(\frac{W_{t+1}}{W_t}\right) + \frac{\beta_i(1+\eta_1(t))}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\log\left(\frac{R_{t+1}}{R_t}\right)\\ &+\frac{1}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\log\left(\frac{H_{t+1}}{H_t}\right) + \frac{\dot{\eta}_2(t)}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\log S_t\\ &+\frac{\dot{\eta}_1(t)}{1-(1-\alpha_i-\beta_i)(1+\eta_1(t))}\log P^i_t + \varepsilon^i_{t+1},\end{aligned} \tag{12}$$

where $\{\varepsilon^i_{t+1}\}$ are random errors from the log-difference approximation. Note that all the coefficients in (12) are "time-varying" because of $F_i(t)$, $A_i(t)$ and the time-varying elasticities $\eta_1(t)$ and $\eta_2(t)$. Moreover, $\log S_t$ and $\log P^i_t$ in (12) are clearly "non-stationary" given such strong tendencies observed in the U.S. real GDP and the sector price indexes (see Fig. 1). Thus, the previously introduced local linear method for non-stationary regressors can be readily employed to estimate the time-varying coefficients of Eq. (12). In the local linear estimation setting, we can rewrite (12) as (13) using time-varying $\beta_i(t)$'s

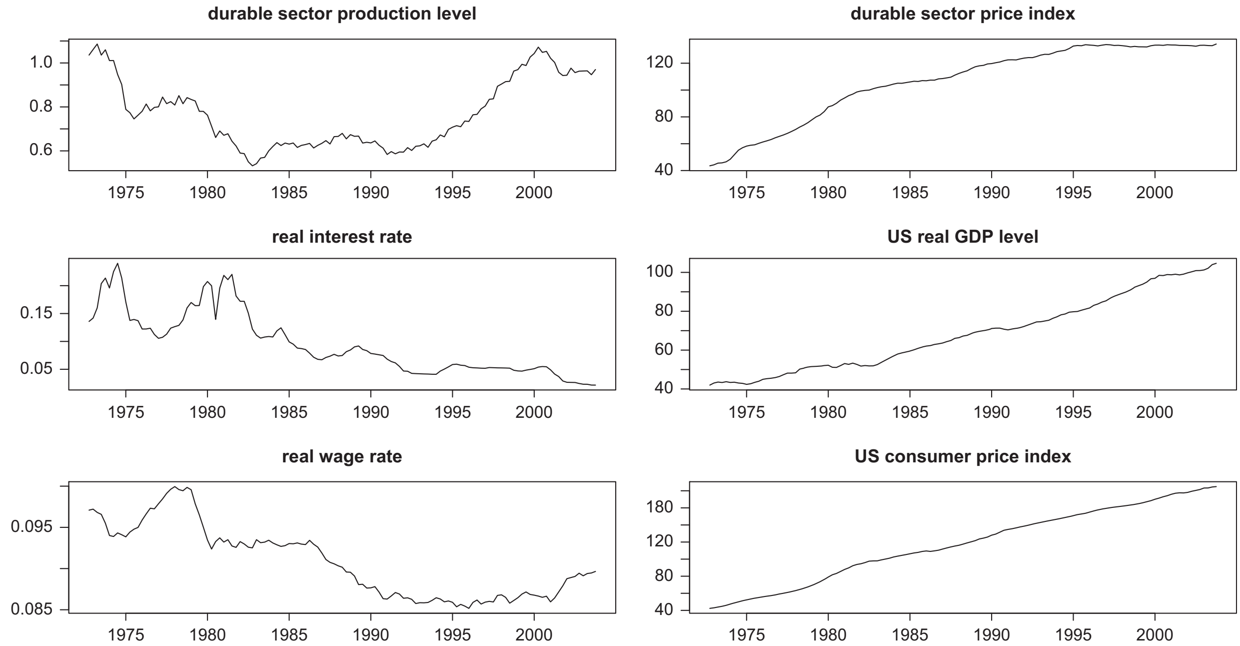

Fig. 1. Time series plots of the macroeconomic variables used in model (13). Quarterly data from January 1972 to March 2004.

that correspond to each coefficient in (12):

$$\log\left(\frac{S^i_{t+1}}{S^i_t}\right) = \beta_0(t)+\beta_1(t)S^i_t+\beta_2(t)\log\left(\frac{S_{t+1}}{S_t}\right)+\beta_3(t)\log\left(\frac{W_{t+1}}{W_t}\right)+\beta_4(t)\log\left(\frac{R_{t+1}}{R_t}\right)$$
$$+\beta_5(t)\log\left(\frac{H_{t+1}}{H_t}\right)+\beta_6(t)\log S_t+\beta_7(t)\log P^i_t+\varepsilon^i_{t+1}, \quad (13)$$

where ε^i_{t+1} can be viewed as the model error term in (1). Note here that Eq. (13) follows within the framework of (1) that we introduced earlier. The bootstrap specification test can be used to test the hypothesis of time-varying coefficient $\beta_i(t)$'s in (13). Among the time-varying coefficients of Eq. (13), $\beta_6(t)=\dot{\eta}_2(t)/(1-(1-\alpha_i-\beta_i)(1+\eta_1(t)))$ and $\beta_7(t)=\dot{\eta}_1(t)/(1-(1-\alpha_i-\beta_i)(1+\eta_1(t)))$ are particularly important since we can verify our claim that both η_1 and η_2 in (8) are time-varying, by testing whether $\beta_6(t)$ and $\beta_7(t)$ are zero or not.

4. Bootstrap specification test

For model (1), an interesting problem is to test parameter constancy. For example, in (13), we want to test $H_1:\beta_k(t)=c_k$, k=0,1,…,7, where c_k's are unknown constants. In particular, we test $H_2:\beta_6(t)=0$ and $H_3:\beta_7(t)=0$ in (13) to test the constancy of the income and price elasticities. Generally, let $L=\{l_1,l_2,\ldots,l_s\}$, $l_1<l_2<\cdots<l_s$, be a subset of $\{1,2,\ldots,p\}$. Denote by $\boldsymbol{\beta}_L(\cdot)$ the function $(\beta_{l_1}(\cdot),\beta_{l_2}(\cdot),\ldots,\beta_{l_s}(\cdot))^\top$. We shall propose a test of the hypothesis $H_L:\boldsymbol{\beta}_L(\cdot)=\mathbf{C}$ of (1) for some constant vector $\mathbf{C}$ of length s. Here $\mathbf{C}$ can be either specified or unspecified. The test is based on comparison of residual sum of squares (RSS) between the null model (RSS_L) and the full model (1) (RSS_F). Similar as ANOVA in multiple linear regression, the idea behind this test is that the aforementioned two RSS's should be close if the null hypothesis is true. In this sense, the test statistic

$$\mathcal{T}=\mathrm{RSS}_L/\mathrm{RSS}_F-1$$

can be used with larger value of $\mathcal{T}$ indicating stronger evidence against the null hypothesis. Let L^c be the complement of L and $\boldsymbol{\beta}_{L^c}(\cdot)$ be defined analogously to $\boldsymbol{\beta}_L(\cdot)$. When $\mathbf{C}$ is specified, then one can plug in the hypothesized value of $\boldsymbol{\beta}_L(\cdot)$ into (1) to obtain

$$y_i-\mathbf{x}_{i,L}^\top\mathbf{C}=\mathbf{x}_{i,L^c}^\top\boldsymbol{\beta}_{L^c}(i/n)+\varepsilon_i,$$

where $\mathbf{x}_{i,L}=(x_{i,l_1},x_{i,l_2},\ldots,x_{i,l_s})^\top$ and $\mathbf{x}_{i,L^c}$ is defined analogously. Given time-varying coefficient models, we use the local linear regression to obtain estimate of $\boldsymbol{\beta}_{L^c}(\cdot)$ and residuals $\hat{\varepsilon}_i=y_i-\mathbf{x}_{i,L}^\top\mathbf{C}-\mathbf{x}_{i,L^c}^\top\hat{\boldsymbol{\beta}}_{L^c}(i/n)$. On the other hand, if $\mathbf{C}$ is unspecified, then model (1) becomes a semiparametric partial linear model unless s=p. If $s<p$, profile least squares method (Fan and Huang, 2005) can be employed to obtain estimates of the coefficients $\hat{\boldsymbol{\beta}}_L$, $\hat{\boldsymbol{\beta}}_{L^c}(\cdot)$ and the residuals $\hat{\varepsilon}_i$. If s=p, then under the null hypothesis, model

(1) is a multiple linear regression model and the classic least squares method can be adopted. In summary, the following bootstrap procedures are used.

(1) Select a bandwidth b_n^* according to the procedures given in Section 6.
(2) Fit the null model with b_n^* to obtain $\hat{\boldsymbol{\beta}}_L$ and $\hat{\boldsymbol{\beta}}_{L^c}(\cdot)$. Note $\hat{\boldsymbol{\beta}}_L = \mathbf{C}$ if $\mathbf{C}$ is specified.
(3) Fit model (1) with b_n^* and obtain the residuals $\hat{\varepsilon}_i = y_i - \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(i/n)$, where $\hat{\boldsymbol{\beta}}(\cdot)$ is the estimated coefficient function under the full model. Calculate the test statistic $\mathcal{T}_0$.
(4) Obtain bootstrap residuals $\varepsilon_i^* = \hat{\varepsilon}_i \vartheta_i$, where ϑ_i are iid standard normal random variables.
(5) Obtain the bootstrap sample $y_i^* = \mathbf{x}_{i,L}^\top \hat{\boldsymbol{\beta}}_L + \mathbf{x}_{i,L^c}^\top \hat{\boldsymbol{\beta}}_{L^c}(i/n) + \varepsilon_i^*$.
(6) Calculate $\mathcal{T}$ for the above bootstrap sample.
(7) Repeat (4)–(6) m (say, 1000) times and obtain m test statistics $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$.
(8) The p-value for H_L can be estimated as $\sum_{i=1}^m I\{\mathcal{T}_i \geq \mathcal{T}_0\}/m$.

Note that the bootstrap samples are generated under the null model while at the same time the bootstrap residuals preserve the dependence structure of the errors. Generalization of the above testing procedure to the hypothesis $\boldsymbol{\beta}_L(\cdot) = \mathbf{P}(t)$, where $\mathbf{P}(t)$ is a specified or unspecified polynomial of t, is straightforward. We notice that similar testing procedures are used in Cai (2007) and our bootstrap test is closely related to the generalized likelihood ratio test of Fan et al. (2001) and Fan and Jiang (2007). For independent samples, the generalized likelihood test has numerous advantages such as independence of nuisance parameters, powerfulness and ease of implementation (Fan and Jiang, 2007). See also Cai (2007) for more discussions on the above testing procedure.

5. Data

The majority of the data in this study were obtained from the Global Insight Basic Economics database at the Wharton Research Data Services (WRDS) of the Wharton School, the University of Pennsylvania (http://wrds.wharton.upenn.edu). The industrial production data were obtained for the "durable" manufacturing section of the U.S. economy, according to the North American Industry Classification System (NAICS). The U.S. industrial production index (IPN) and the U.S. nominal GDP series were converted to real values by dividing the nominal values by the U.S. implicit GDP deflator. Quarterly data were obtained by taking the quarterly averages of the monthly values of each year from January 1972 to March 2004.

The U.S. Consumer Price Index (CPI) and the wage rate series were taken from the Department of Labor, Bureau of Labor Statistics (BLS), at http://stats.bls.gov. The wage rates were converted into real ones by dividing the nominal values by the CPI. The price index data for the "durable" manufacturing section of the U.S. economy were taken from the Global Insight Economic Database (Monthly Series) at the WRDS. The monthly price data were available from January 1972. The quarterly averages were taken. The monthly series of the U.S. prime interest rate from January 1972 until March 2004 were taken from the Federal Reserve System (Board of Governors), at http://www.federalreserve.gov. The series were converted into real values by dividing the nominal values by the U.S. Consumer Price Index (CPI). The data for the U.S. real GDP term are available on the quarterly basis, and we thus work with the "quarterly" data of 126 observations for each variable in model (13). In Fig. 1, we plot the data for some of the variables in (13).

6. Empirical results

In order to run the local linear regression on (13), we first choose the optimal "bandwidth" under the generalized cross-validation (GCV) method (Wahba, 1977; Craven and Wahba, 1979). By (5), we can write the predicted values

$$M(b) = H(b)Y = H(b)(y_1, \ldots, y_n)^\top, \tag{14}$$

where $M(b) := (m_1(b), \ldots, m_n(b))^\top = (\mathbf{x}_1^\top \hat{\boldsymbol{\beta}}_b(1/n), \ldots, \mathbf{x}_n^\top \hat{\boldsymbol{\beta}}_b(n/n))^\top$ and $H(b)$ is a $(n \times n)$ smoothing matrix. The GCV criterion chooses the bandwidth b_n^* that minimizes

$$\mathrm{GCV}(b) = \frac{n^{-1} \sum_{i=1}^n (y_i - m_i(b))^2}{\{1 - \mathrm{trace}[H(b)]/n\}^2}. \tag{15}$$

The optimal bandwidth chosen by the GCV balances the goodness-of-fit represented by the numerator of (15) and model complexity measured in the denominator. In fact, the quantity trace$[H(b)]$ can be viewed as degrees of freedom of model (1) with bandwidth b (Hastie and Tibshirani, 1993). The criterion has nice properties such as easy implementation and avoiding the estimation of nuisance parameters in the model. For more on the GCV, see Golub et al. (1979), Li (1985), Wahba (1990) and Fan and Jiang (2007). The GCV chooses bandwidth b_n^*=0.36 (see Fig. 2), which will be used for both estimation and hypothesis testings. Note here that a different bandwidth can be used for hypothesis testings.

Given the bandwidth b_n^*=0.36, we estimate (13) using the local linear regression. Data from the "durable" manufacturing section of U.S. economy along with other U.S. macroeconomic data are used for the estimation. The estimated coefficient functions $\beta_i(t)$, i=0,1,...,7, are summarized in Fig. 3. It appears clear from the empirical results that all

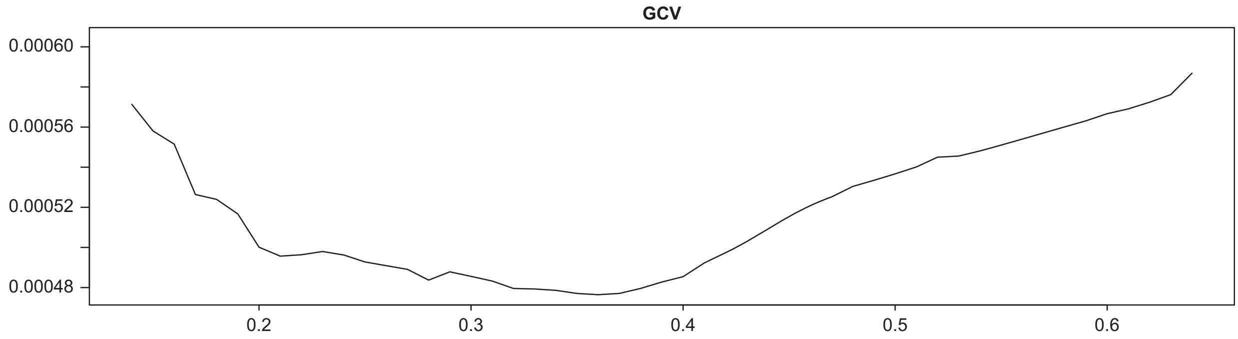

Fig. 2. Plot of the GCV statistic with respect to bandwidth for model (13).

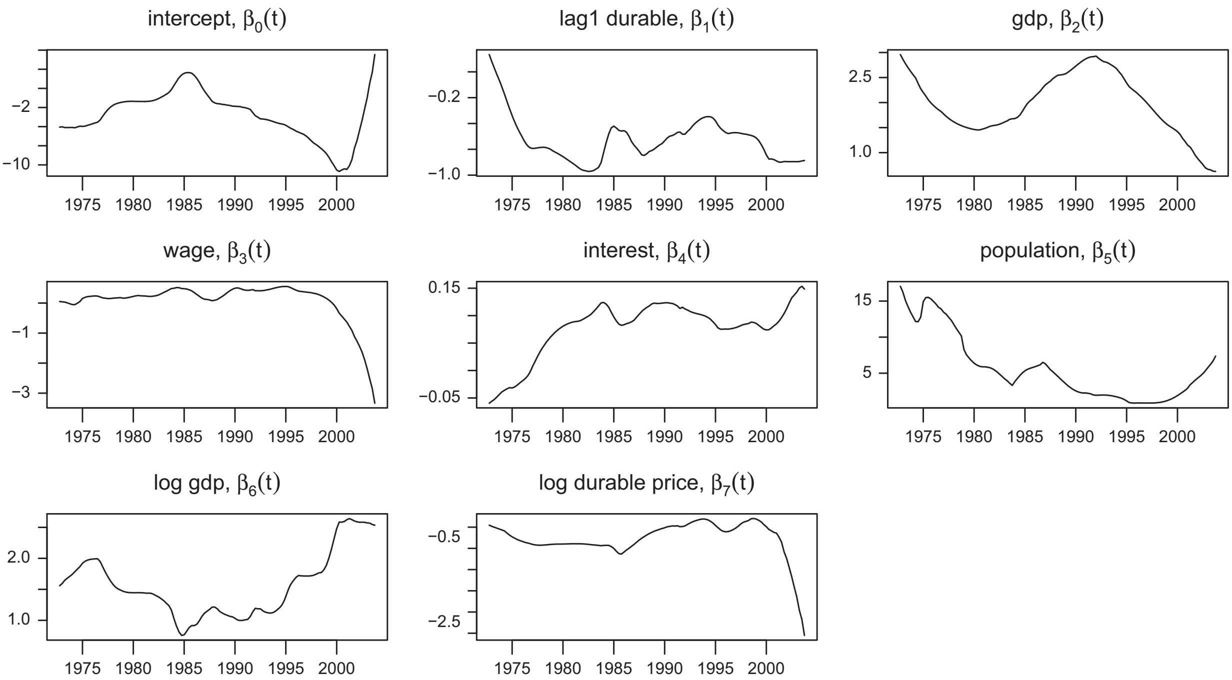

Fig. 3. Plot of estimated coefficient functions of model (13).

Table 1
Bootstrap test results for the hypotheses H_1 to H_3.

Hypothesis	Test statistic	95% cut off value	p-Value
H_1	0.849	0.344	<0.001
H_2	0.369	0.126	<0.001
H_3	0.255	0.093	<0.001

Cut-off and p-value are based on 1000 bootstrap samples.

the model coefficient functions in (13) are "time-varying". This suggests that the use of time-varying coefficient model (13) is necessary. More importantly, the coefficient functions $\beta_6(t)$ and $\beta_7(t)$ seem significantly different from 0, indicating that the price and the income elasticities of demand (η_1 and η_2, respectively) are indeed "time-varying" as we assumed in derivation of Eq. (13).

We shall employ the bootstrap specification tests in order to justify our claims. Here 1000 bootstrap samples are generated to test each of the following five hypotheses: $H_1 : \beta_k(t) = c_k$, k=0,1,…,7 in (13); $H_2 : \beta_6(t) = 0$ and $H_3 : \beta_7(t) = 0$. where c_k's k=0,1,…,7 are unknown constants. The results of the bootstrap test are summarized in Table 1. Given the extremely small p-values (<0.001), we have a strong evidence "against" H_1, H_2 and H_3. The rejection of H_1 shows the necessity of time-varying coefficient model, and the rejection of H_2 and H_3 justifies the assumption of time-varying elasticities of demand.

Among the test results here, the rejection of $H_3 : \beta_7(t) = 0$ is particularly interesting because it tells us a meaningful pattern of change in the price elasticity of demand for the U.S. durable goods market. Given that $\beta_7(t)$ is mostly "negative" from Fig. 3, the rejection of $H_3 : \beta_7(t) = 0$ means that $\dot{\eta}_1(t)/\eta_1(t)$ is mostly "positive" under the initial assumptions $0 < \alpha_i + \beta_i < 1$ and $\eta_1(t) < 0$. This suggests that $\eta_1(t)$ has been "increasing in its absolute value" over the years, which means that consumers in the U.S. durable goods market have been more and more "price-elastic" over the years. Hackl and Westlund (1995, 1996) also report similar outcomes based on their estimation of time-varying price elasticities for the European telecommunication demands. They find out that the absolute price elasticities for telecommunication demands in Germany, the UK and the Scandinavian countries have increased over the years and conclude that the assumption of time-invariant price elasticity is not appropriate when observations over a long time period are used to fit their demand function.

This interesting phenomenon can be explained in part by the following: Consumers in modern days have easier and faster access to a large amount of "market information" through advanced technologies such as telephones or the internet in particular. For example, consumers in the automobile (i.e. durable good) market can easily compare different prices on the same vehicle offered by different dealerships from the internet. As a result, they tend to react more sensitively to the same change in the vehicle price than they did before. It means that such fundamental changes in market over the years have made consumers more and more "price-sensitive", which appears to be what we observe in our empirical study here.

In fact, market-specific factors such as technological advances or pricing strategy of firms can cause structural changes in the market, which can lead to changes in demand elasticities. Technological developments are usually rapid for the durable manufacturing section of the economy, such as computer or automobile industries, and their impact is generally bigger than that for the other sections. Given such characteristics of the durable goods market and the cause and the effect

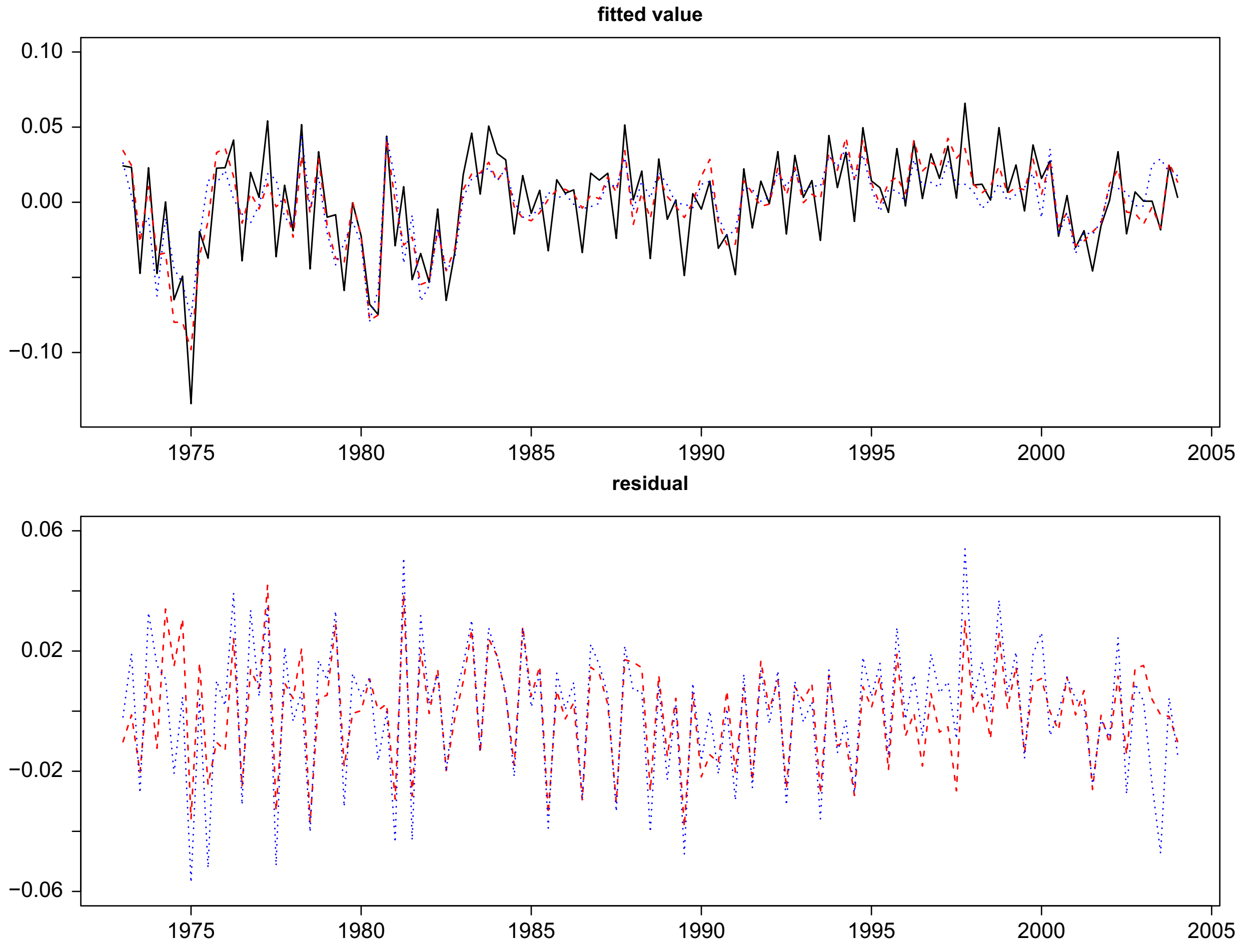

Fig. 4. Comparison of prediction performance and residuals of models (13) and MLM. Upper panel: plot of predicted value of model (13) (dashed line) and MLM (dotted line), along with the plot of the response (solid line). Lower panel: residual plots of model (13) (dashed line) and MLM (dotted line).

of structural changes, we expect the demand elasticities of the durable goods market to vary over time, which is verified by the empirical evidence here.

We now investigate the "in-sample" prediction property of the local linear estimator, compared to the multiple linear model (MLM) where all coefficients are time-invariant. Fig. 4 summarizes the predicted (fitted) values and residuals of (13) and MLM. It is clear from the prediction graphs that dynamics of the response are well represented by the local linear model, particularly in terms of capturing the local fluctuations of the response. Moreover, from the residuals plot, we see that the residuals of (13) are much less volatile than those of the MLM. In fact, from the first row of Table 1, we see that the RSS of (13) with bandwidth 0.36 is about $\frac{1}{1+0.849} \approx 54\%$ of that from MLM. The 46% gain in RSS for the time-varying coefficient model looks impressive.

7. Concluding remarks

The comparison of the local linear model and the MLM in Fig. 4 clearly illustrates why we can benefit from using time-varying coefficient models in economic model building. As mentioned in the Introduction, structural change is a common feature in the economy (Phillips, 2001). Thus, given such data and models, it makes much more sense to adopt the model with time-varying coefficients (TVC) rather than the one with time-invariant parameters, because a model with TVC inherently picks up the structural changes of the economy well. That is why the RSS is reduced significantly when we employ the local linear model instead of the MLM, as illustrated in Fig. 4.

Our economic-causal model (13) is grounded in economic theory, and it works empirically much better than the fixed-coefficient counterpart as we have observed. The local linear regression and the bootstrap specification test validate the time-varying coefficient model, and the key assumption of time-varying demand elasticities are also confirmed. Demand elasticities are affected by market-specific factors such as technological developments. For the empirical study, we choose the durable manufacturing section of the U.S. economy to test the hypothesis of time-varying elasticities because technological change is usually more rapid for the durable goods industry than for other sections of the economy. The empirical results confirm our claim.

Appendix A. Asymptotic results

A.1. Dependence measures

Appropriate dependence measures are needed in asymptotic analysis of the estimates of regression function $\boldsymbol{\beta}(\cdot)$. In this section we shall introduce "physical" dependence measures of non-stationary systems. Let $(\varepsilon_i^*)_{i\in\mathbb{Z}}$ be an iid copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$. Define the coupled process $\mathcal{F}_i^* = (\ldots,\varepsilon_0^*,\ldots,\varepsilon_i)$, i=0,1,.... Recall $\mathcal{F}_i = (\ldots,\varepsilon_{i-1},\varepsilon_i)$. For a vector $\mathbf{v} = (v_1,v_2,\ldots,v_p) \in \mathbb{R}^p$, let $|\mathbf{v}| = (\sum_{i=1}^p v_i^2)^{1/2}$. For a random vector $\mathbf{V}$, write $\mathbf{V} \in \mathcal{L}^q$ $(q>0)$ if $\|\mathbf{V}\|_q := [\mathbb{E}(|\mathbf{V}|^q)]^{1/q} < \infty$ and $\|\mathbf{V}\| = \|\mathbf{V}\|_2$. We define physical dependence measures for $\mathbf{G}(t,\mathcal{F}_i)$ as

$$\delta(\mathbf{G},k,q) = \sup_{t\in[0,1]} \|\mathbf{G}(t,\mathcal{F}_k) - \mathbf{G}(t,\mathcal{F}_k^*)\|_q, \quad k \geq 0. \tag{16}$$

We can also define dependence measures $\delta(H,k,q)$ of the system $H(t,\mathcal{G}_i)$ in the same way. Wu (2005) introduced physical dependence measures for stationary causal processes. With $\delta(\mathbf{G},k,q)$ and $\delta(H,k,q)$, we are able to establish an asymptotic theory for local linear estimates of $\beta(\cdot)$.

Apparently, $\delta(\mathbf{G},k,q)$ quantifies the overall dependence of $\mathbf{G}(t,\mathcal{F}_k)$ on the innovation ε_0 by measuring the $\mathcal{L}^q$ distance between $\mathbf{G}(t,\mathcal{F}_k)$ and its coupled version $\mathbf{G}(t,\mathcal{F}_k^*)$. Note if $G(t,\mathcal{F}_k)$ is functionally unrelated to ε_0, $\delta(\mathbf{G},k,q)$ is 0. The above dependence measures are closely related to the data generating mechanism and hence easy to deal with. See Zhou and Wu (2009) for calculations of $\delta(\mathbf{G},k)$ for locally stationary linear processes and a general class of locally stationary nonlinear processes.

A.2. Convergence rate of the local linear estimator

We shall first establish consistency and convergence rate of the local linear estimator. To this end, we shall introduce some notation and regularity conditions. Denote by $\mathbf{S}(t,\mathcal{A}_i)$ a generic locally stationary system. Let $\mathcal{U}_q([0,1])$ be the collection of systems such that $\sup_{0\le t\le 1}\|\mathbf{S}(t,\mathcal{A}_i)\|_q < \infty$.

(A1) Let $M(t)$ be the $p\times p$ matrix with ijth entry $m_{ij}(t) = \mathbb{E}[G_i(t,\mathcal{F}_0)G_j(t,\mathcal{F}_0)]$. Assume that the smallest eigenvalue of $M(t)$ is bounded away from 0 on [0,1]. Furthermore, $m_{ij}(\cdot)$ is Lipschitz continuous on [0,1], i,j=1,2,...,p.
(A2) $\mathbf{G}(t,\mathcal{F}_i) \in \mathcal{U}_4([0,1])$.
(A3) $\mathbf{U}(t,\mathcal{D}_i) \in \mathcal{U}_2([0,1])$, where $\mathbf{U}(t,\mathcal{D}_i) = \mathbf{G}(t,\mathcal{F}_i)H(t,\mathcal{G}_i)$.
(A4) $\sum_{k=0}^{\infty}[\delta(\mathbf{G},k,4)+\delta(\mathbf{U},k,2)] < \infty$.

A few remarks on the regularity conditions are in order. Eigenvalue constraint in condition (A1) prevents asymptotic multicollinearity of the explanatory variables and is thus clearly necessary. It is a common practice to assume bounded fourth moment of the predictors for model (1); see for instance Cai (2007) and Orbe et al. (2005), which in fact posed stronger moment assumptions. Lipschitz continuity assumption in condition (A1) mean local stationarity and it asserts smoothness of functions **G** with respect to t. Finally, condition (A4) asserts that the cumulative contributions of innovations η_0 and ε_0 in generating future outcomes of **G** and **U** is uniformly finite and hence implies short-range dependence of the series. When **G** and **U** are stationary linear processes, condition (A4) is equivalent to the stability (short range dependent) condition of Box and Jenkins (1976). We refer the readers to Zhou and Wu (2009) for discussions of validity of condition (A4) for general locally stationary time series models.

Theorem 1. *Assume* (A1)–(A4), $b_n \to 0$ *and* $nb_n \to \infty$. *Then for any fixed* $t \in (0,1)$

$$\|\hat{\boldsymbol{\beta}}(t)-\boldsymbol{\beta}(t)\| = O\left(b_n^2 + \frac{1}{\sqrt{nb_n}}\right). \tag{17}$$

Theorem 1 provides a convergence rate of the local linear estimator under natural conditions on the bandwidth b_n and mild conditions (A1)–(A4). Eq. (17) implies that $\hat{\boldsymbol{\beta}}(t)$ is a consistent estimator of $\boldsymbol{\beta}(t)$ with asymptotic mean squared error (MSE) $b_n^4 + 1/nb_n$. Therefore, the bandwidth which minimizes the MSE should be of order $n^{-1/5}$. This is consistent with the classic results on the local liner estimators (Fan and Gijbels, 1996).

First, we let C denote a finite constant which may vary from line to line. Moreover, let $\mathcal{A}_i = (\ldots, \eta_{i-1}, \eta_i)$ with η_i's (*iid*) and $S(t,\mathcal{A}_i) : [0,1] \times \mathbb{R}^\infty \to \mathbb{R}$ be a locally stationary system such that $S(t,\mathcal{A}_i) \in \mathcal{U}_2([0,1])$. Denote by $\delta(S,k,p)$ physical dependence measures of S similar as those defined in (16). Define time series $D_i = S(i/n,\mathcal{A}_i)$. Let $\Psi_n(t,r) = \sum_{i=1}^n D_i w(t,i,r)$, where $w(t,i,r) = [(i/n-t)/b_n]^r K_{b_n}(i/n-t)/(nb_n)$.

Lemma 1. *Assume* $\sum_{k=0}^\infty \delta(S,k,2) < \infty$, $b_n \to 0$ *and* $nb_n \to \infty$. *Then*

$$\|\Psi_n(t,r) - \mathbb{E}\Psi_n(t,r)\| = O\left(\frac{1}{\sqrt{nb_n}}\right).$$

Proof. For $k \in \mathbb{Z}$ define the projection operator

$$\mathcal{P}_k(\cdot) = \mathbb{E}(\cdot|\mathcal{A}_k) - \mathbb{E}(\cdot|\mathcal{A}_{k-1}). \tag{18}$$

Let $\Psi_{n,k}(t,r) = \sum_{i=1}^n \mathcal{P}_{i-k} D_i w(t,i,r)$, k=0,1,.... Note that the summands of $\Psi_{n,k}(t,r)$ form a martingale difference sequence. Therefore

$$\begin{aligned}\|\Psi_{n,k}(t,r)\|^2 &= \sum_{i=1}^n \|\mathcal{P}_{i-k}D_i\|^2 w^2(t,i,r) \le \sum_{i=1}^n \|D_i - D_{i,k}^*\|^2 w^2(t,i,r) \\ &\le \sum_{i=1}^n \delta(S,k,2)^2 w^2(t,i,r) \le C\delta(S,k,2)^2/(nb_n),\end{aligned}$$

where $D_{i,k}^* = S(i/n,\mathcal{A}_{i,k}^*)$ with $A_{i,k}^* = (\ldots,\eta_{i-k-1},\eta_{i-k}^*,\eta_{i-k+1},\ldots,\eta_i)$ and $\{\eta_i^*\}$ an iid copy of $\{\eta_i\}$. Note $\Psi_n(t,r) - \mathbb{E}\Psi_n(t,r) = \sum_{k=0}^\infty \Psi_{n,k}(t,r)$. Hence

$$\|\Psi_n(t,r) - \mathbb{E}\Psi_n(t,r)\| \le C\sum_{k=0}^\infty \delta(S,k,2)/(\sqrt{nb_n}) = O(1/\sqrt{nb_n}).$$

The lemma then follows. □

Lemma 2. *Let* $\varpi(t) = \mathbb{E}S(t,\mathcal{A}_0)$. *Assume that* $\varpi(t)$ *is Lipschitz continuous on* $[0,1]$. *Also, let* $b_n \to 0$ *and* $nb_n \to \infty$. *Then*

$$\mathbb{E}\Psi_n(t,r) - \mu_i \varpi(t) = O\left(b_n + \frac{1}{nb_n}\right).$$

Proof. By the Lipschitz continuity of $\varpi(t)$. We have

$$\begin{aligned}\mathbb{E}\Psi_n(t,r) &= \sum_{i=1}^n \mathbb{E}D_i w(t,i,r) = \sum_{i=1}^n \varpi(i/n) w(t,i,r) \\ &= \sum_{i=1}^n [\varpi(t) + O(b_n)] w(t,i,r) \\ &= \mu_i \varpi(t) + O\left(b_n + \frac{1}{nb_n}\right).\end{aligned}$$

This lemma follows. □

Proof of Theorem 1. By (5) and Taylor's expansion of $\boldsymbol{\beta}(t)$, we have

$$\mathbf{S}_n(t)(\hat{\boldsymbol{\eta}}(t)-\boldsymbol{\eta}(t)) = \begin{pmatrix} b_n^2\mathbf{S}_{n,2}(t)(\boldsymbol{\beta}''(t)+o(1))/2 \\ b_n^2\mathbf{S}_{n,3}(t)(\boldsymbol{\beta}''(t)+o(1))/2 \end{pmatrix} + \begin{pmatrix} \mathbf{T}_{n,0}(t) \\ \mathbf{T}_{n,1}(t) \end{pmatrix}, \tag{19}$$

where $\boldsymbol{\eta}(t) = (\boldsymbol{\beta}^\top(t),(\boldsymbol{\beta}'(t))^\top)^\top$ and $\mathbf{T}_{n,i}(t) = (nb_n)^{-1}\sum_{j=1}^n \mathbf{x}_j\varepsilon_j[(t_j-t)/b_n]^i K_{b_n}(t_j-t)$ for i=0,1. Write $\mathbf{T}_n(t) = (\mathbf{T}_{n,0}(t),\mathbf{T}_{n,1}(t))^\top$. Using Schwarz's inequality, by (A2) and (A4),

$$\sum_{k=0}^{\infty} \delta(G_i(t,\mathcal{F}_s)G_j(t,\mathcal{F}_s),k,2) < \infty, \quad i,j = 1,2,\ldots,p.$$

By Lemmas 1 and 2 and the fact that $\mathbb{E}\mathbf{T}_{n,j}(t) = 0$, j=0,1, we have

$$|\mathbf{S}_{n,i}(t)-\mu_i M(t)| = O_{\mathbb{P}}((nb_n)^{-1/2}+b_n), \quad i = 0,\ldots,3,$$

$$|\mathbf{T}_{n,j}(t)| = O_{\mathbb{P}}((nb_n)^{-1/2}), \quad j = 0,1.$$

By (19), since $M(t)$ is non-singular, Theorem 1 follows. □

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