The Poisson Distribution.

There are two basic types of probability distributions:

1. **Discrete**
   
   (a) \( X \sim \text{Bernoulli}(p) \)
   
   This means that \( X = \begin{cases} 1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-p \end{cases} \)
   
   (b) \( X \sim \text{Binomial}(n, p) \), \( n = \text{# of trials} \)
   
   \[ P(X) = \binom{n}{x} p^x (1-p)^{n-x} \], \( x = 0, 1, 2, \ldots \)
   
   (c) \( X \sim \text{Uniform}(N) \), \( N = \text{# of trials} \)
   
   \[ P(X) = \frac{1}{N} \]

   E.g., roll a die, \( X = \# \text{ on the upper faces} \)
   
   \[ P(X) = \frac{1}{6} \], \( x = 1, 2, 3, 4, 5, 6 \)

2. **Continuous**

   (a) \( X \sim \text{Normal}(\mu, \sigma^2) \)
   
   \[ f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \], \( -\infty < x < \infty \)
   
   \[ P(a \leq X \leq b) = \int_a^b f(x) \, dx \]

   (b) \( X \sim \text{Uniform}(a, b) \)
   
   \[ f(x) = \frac{1}{b-a} \], \( x \in (a, b) \)

   \[ 0 \leq \frac{1}{a+x} \leq 1 \]
We'll introduce one more important discrete distribution, called the Poisson Distribution.

It is very similar to the Binomial dist'n.

The Binomial is the dist'n for a 'counting' type of variable, where we are counting the number of times some event (a success) occurs over a fixed number of 'trials'. Events occur at a constant rate (specifed by \( p \)), and these events are independent.

E.g. Flip 10 coins and count heads.

\[ \text{H T H T H H T T} \quad X = \# \text{of heads} = 6 \]

Change the fixed number of trials to a fixed amount of time, and count events.

<table>
<thead>
<tr>
<th>1:00pm</th>
<th>2:30</th>
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</table>

\[ X = \# \text{of customers} = 6 \]

It may be shown that the dist'n of \( X = \# \text{of events} \) occurring over \( t \) units of time is

\[ P(X) = \frac{e^{-\lambda t} (\lambda t)^x}{x!} \text{ for } x=0,1,2,... \]

where \( \lambda \) is the average number of events occurring in one unit of time.

\[ n! = 1, 2, 3, ..., (n-1)n \quad 0! = 1 \]
E.g. If customers arrive at the rate of 5 per hour on average, then the probability that 6 customers will arrive in the 1.5 hour period 2-3:30 pm is:

\[ P(6) = \frac{e^{-\lambda t} (\lambda t)^6}{6!} = 0.137 \]

\[ \lambda = 5, \ t = 1.5 \]

Assumptions:
1. Events occur independently of each other.
2. There is a constant rate of occurrence of events over the time interval, i.e., the probability of an event occurring at time \( t_1 \) is the same as the probability of an occurrence at time \( t_2 \).

It can be shown that if \( X \sim \text{Poisson}(\lambda t) \), then

\[ \mu = E(X) = \lambda t \]

\[ \sigma^2 = \text{Var}(X) = \lambda t \]

First, let's show that \[ \sum_{x=0}^{\infty} p(x) = 1 \], where

\[ p(x) = \frac{e^{-\lambda t} (\lambda t)^x}{x!} \]

\[ \sum_{x=0}^{\infty} e^{-\lambda t} (\lambda t)^x = e^{-\lambda t} \sum_{x=0}^{\infty} (\lambda t)^x = e^{-\lambda t} \left( \frac{e^\lambda}{1 - t} \right) = e^{-\lambda t + \lambda t} = e^0 = 1 \]
Theorem. If \( X \sim \text{Poisson}(\lambda t) \), then \( \mu = E(X) = \lambda t \) and \( \sigma^2 = \text{Var}(X) = \lambda t \).

Proof:

\[
E(X) = \sum_{x=0}^{\infty} x \cdot p(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda t} \lambda^x t^x}{x!} \frac{e^{-\lambda t} \lambda^x t^x}{x!}
\]

\[
= \sum_{x=1}^{\infty} \frac{e^{-\lambda t} \lambda^x t^x}{(x-1)!} \frac{e^{-\lambda t} \lambda^x t^x}{x!} = (z = x-1)
\]

\[
= (\lambda t) \sum_{z=0}^{\infty} \frac{e^{-\lambda t} \lambda^{z+1} t^{z+1}}{z!(z+1)!} = \lambda t
\]

\[
\text{Var}(X) = E(X^2) - E(X)^2 = \frac{E(X(X-1))}{E(X)} + E(X) - E(X)^2
\]

Try to finish at home.

So the mean and variance must be exactly equal to each other. When we specify a particular Poisson distribution, the mean is the parameter we must specify, so Poisson(3) indicates a Poisson distribution with \( \mu = \lambda t = 3 \).

E.g. the Poisson distribution with \( \mu = 3 \) looks like:
Scatterplot of prob vs x

\[ p(0) = e^{-3} \frac{3^0}{0!} \]
\[ p(1) = e^{-3} \frac{3^1}{1!} \]

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Approximating Binomial Probabilities

Binomial dist'n formula:
\[ P(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad n \leq 5 \]

If \( p \) is small (\( \leq 0.05 \)), and \( n \) is not too small, we may replace Binomial by Poisson with \( \mu = np \):
\[ P(x) = \frac{e^{-\mu} \mu^x}{x!} \]

We can show:
\[ \lim_{n \to \infty} \binom{n}{x} p^x (1-p)^{n-x} = \frac{e^{-\lambda t} (\lambda t)^x}{x!} , \quad \mu = \lambda t = np \]

Proof:
\[ \lim_{n \to \infty} \binom{n}{x} p^x (1-p)^{n-x} = \]
\[ \binom{n}{x} = \frac{n!}{x! (n-x)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-x+1)}{x!} \]
\[ = \frac{n(n-1) \cdot \ldots \cdot (n-x+1)}{x!} \]
\[ \Rightarrow \lim_{n \to \infty} \frac{n(n-1) \cdot \ldots \cdot (n-x+1)}{x!} = \frac{(\lambda t)^x}{x!} \lim_{n \to \infty} \frac{n^x}{x!} \cdot \frac{1}{(1-p)^{(n-x)}} \]
\[ = \frac{(\lambda t)^x}{x!} \lim_{n \to \infty} \frac{n^x}{x!} \cdot \frac{1}{(1-p)^{(n-x)}} \]
\[ = \frac{(\lambda t)^x}{x!} \lim_{n \to \infty} (1 - \frac{\lambda t}{n})^{n-x} \]
\[ = \frac{(\lambda t)^x}{x!} \lim_{n \to \infty} (1 - \frac{\lambda t}{n})^{n-x} \]
Ex. Reactions to a certain drug occur on average for 1 in every 1000 patients. What is the probability that in a group of 1500 patients, no reactions will be observed?

Solution:

\[ X = \# \text{ of reactions} \]
\[ X \sim \text{Bin}(1500, \frac{1}{1000} = 0.001) \]
\[ E(X) = np = 1.5 = \lambda \]
\[ P(X = 0) = p(0) = \frac{e^{-1.5} \cdot (1.5)^0}{0!} = e^{-1.5} = 0.223 \]

Using binomial:
\[ P(X = 0) = p(0) = \binom{1500}{0} 0.001^0 (1 - 0.001)^{1500} = 1 \cdot 1 \cdot 0.999^{1500} = \]

Ex. Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of 7 per hour. During a given hour, what are the probabilities that
(a) no more than 3 customers arrive?
(b) at least 2 customers arrive?
(c) exactly 5 customers arrive?

Solution:

\[ X = 7.1 = \lambda \]

(a) \[ P(X \leq 3) = \sum_{x=0}^{3} p(x) = p(0) + p(1) + p(2) + p(3) \]
\[ = \frac{e^{-\lambda} \cdot \lambda^0}{0!} + \frac{e^{-\lambda} \cdot \lambda^1}{1!} + \frac{e^{-\lambda} \cdot \lambda^2}{2!} + \frac{e^{-\lambda} \cdot \lambda^3}{3!} \]
\[ = 0.0818 \]
(b) \[ P(X \geq 2) = 1 - P(X \leq 2) \]
\[ = 1 - P(X = 1) \]
\[ = 1 - \left[ p(0) + p(1) \right] \]
\[ = 1 - \left( \frac{e^{-\lambda} \cdot \lambda^0}{0!} + \frac{e^{-\lambda} \cdot \lambda^1}{1!} \right) = 0.9927 \]

(c) \[ P(X = 5) = p(5) = \frac{e^{-\lambda} \cdot \lambda^5}{5!} = 0.1277 \]

\( e \approx 2.7 \)