<u>Lecture 9</u> (Techniques of Integration)

Integration by Parts

Let's recall the product rule for differentiation:

$$\int \frac{d}{dx}[f(x)g(x)] = \int f(x)g'(x) + f'(x)g(x)$$

If we take the indefinite integral of both parts, we shall get

$$f(x)\,g(x) = \int f(x)\,g'(x)\,dx + \int f'(x)\,g(x)\,dx$$

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

This gives us the formula for **integration by parts**:

$$\boxed{\int \underbrace{f(x)}_{u}\underbrace{g'(x)dx}_{dv} = f(x)g(x) - \int g(x)f'(x)dx}$$

Very often it is used in the following notations:

Let

$$u = f(x) \quad \text{and} \quad v = g(x)$$

Then

$$du = f'(x)dx \quad \text{and} \quad dv = g'(x)dx$$

Thus,

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: Find $\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = u \cdot v - \int v \, du$

$u = x \qquad dv = \cos x \, dx$

$du = dx \qquad v = \sin x$

$= x \sin x - \int \sin x \, dx$

$= x \sin x + \cos x + C$

Example: Evaluate $\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} = uv - \int v\, du$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$= x \ln x - \int x \cdot \frac{1}{x} dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

<u>Example</u>: Find $\int xe^{-x}dx$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$= uv - \int v\,du$

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} + (-e^{-x}) + C$

$= -e^{-x}(x+1) + C$

Example: Evaluate $\int x^2 e^x dx = uv - \int v\,du$

$u = x^2 \qquad dv = e^x dx$

$du = 2x\,dx \qquad v = e^x$

$$= x^2 e^x - 2\underbrace{\int e^x x\,dx}$$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$= x^2 e^x - 2\left[x e^x - \int e^x dx\right]$$

$$= x^2 e^x - 2\left[x e^x - e^x\right] + C$$

$$= e^x\left[x^2 - 2x - 2\right] + C$$

Example: Evaluate $\int e^x \sin x \, dx$ $= uv - \int v \, du$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x \, dx \qquad v = e^x$

$= e^x \sin x - \int e^x \cos x \, dx$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x dx \qquad v = e^x$

$\int e^x \sin x dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$

$= e^x \sin x - e^x \cos x - \int e^x \sin x dx$

$2 \int e^x \sin x dx = e^x (\sin x - \cos x) + C$

$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$

Finding definite integrals by parts:

$$\boxed{\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx}$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example: $\int_0^1 xe^x dx$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$= xe^x\Big|_0^1 - \int_0^1 e^x dx$$

$$= [1 e^1 - 0 \cdot e^0] - e^x\Big|_0^1$$

$$= e - [e^1 - e^0] = e^0 = 1$$

Example: Calculate $\int_0^1 \tan^{-1} x\, dx$

$u = \tan^{-1} x \qquad dv = dx$

$du = \frac{1}{1+x^2} dx \qquad v = x$

$= x \tan^{-1} x \Big|_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2} \cancel{dx}$

$u = 1 + x^2$

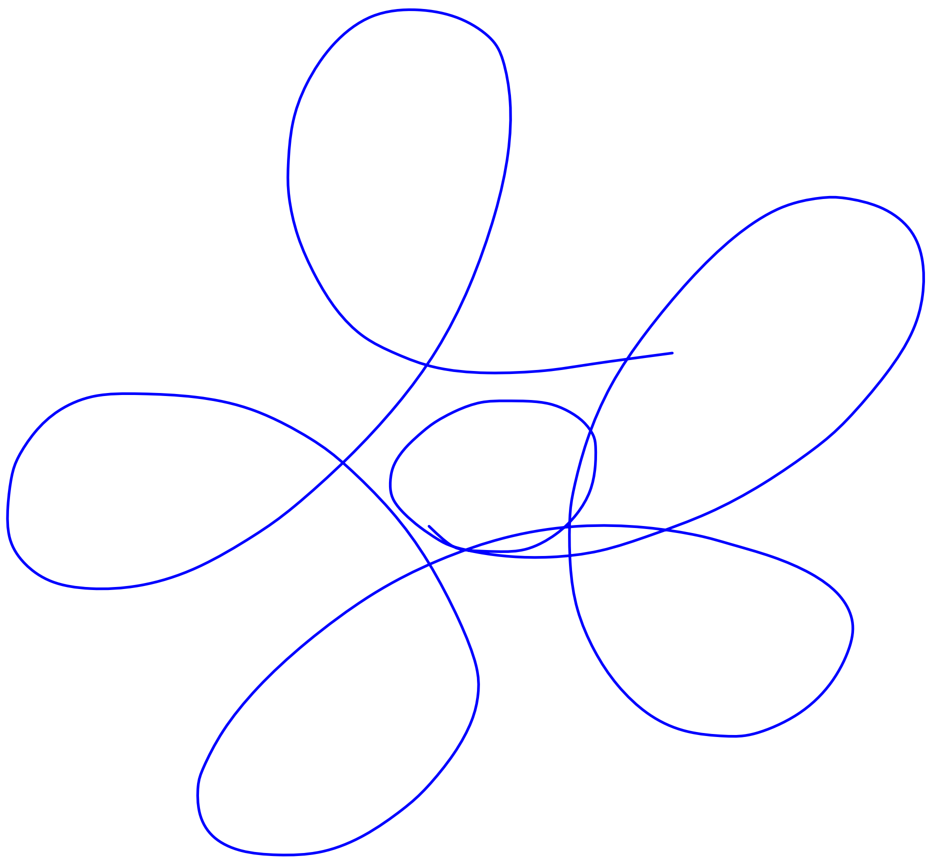
$du = 2x\, dx$

$x\, dx = \frac{1}{2} du$

$= 1 \cdot \frac{\pi}{4} - \frac{1}{2} \int_1^2 \frac{1}{u} du$

$= \frac{\pi}{4} - \frac{1}{2} \ln u \Big|_1^2$

$= \frac{\pi}{4} - \frac{1}{2} [\ln 2 - \ln 1] = \frac{\pi}{4} - \frac{1}{2} \ln 2$



<u>Example</u>: Prove the reduction formula

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$$

and use it to evaluate $\int \cos^2 x\, dx$.

$$\int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$$

$$u = \cos^{n-1} x \qquad dv = \cos x\, dx$$

$$v = \sin x$$

$$du = (n-1)\cos^{n-2} x\,(-\sin x)\,dx$$

$$= \cos^{n-1} x \sin x + \int \sin x\,(n-1)\cos^{n-2} x \sin x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx$$

$$\sin^2 x = 1 - \cos^2 x$$

$$= \cos^{n-1} x \sin x + (n-1)\int \left[\cos^{n-2} x - \cos^n x\right] dx$$

$$\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^{n}x\,dx$$

$$\int \cos^n x\,dx + (n-1)\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$n\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\int \cos^2 x\,dx = \frac{1}{2}\cos x\sin x + \underbrace{\frac{2-1}{2}\int \cos^0 x\,dx}_{\frac{1}{2}\int dx}$$

$n=2$

$$= \frac{1}{2}\cos x\sin x + \frac{1}{2}x + C$$

Trigonometric Integrals

We can often simplify trigonometric functions using the following

Trigonometric identities:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double-angle formulas:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

$$= 2\cos^2 x - 1$$

Half-angle formulas:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \cos^2 2x = \frac{1}{2}(1 + \cos 4x)$$

Example: Evaluate $\int \sin^2 x \, dx$ $= \frac{1}{2}\int (1 - \cos 2x)\, dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x \, dx$

$u = 2x$

$du = 2\, dx$

$\frac{1}{2}du = dx$

$= \frac{1}{2}x - \frac{1}{2}\frac{1}{2}\int \cos u \, du$

$= \frac{1}{2}x - \frac{1}{4}\sin u + C = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

Example: Evaluate $\int \sin^3 x \, dx$ $= \int \sin^2 x \sin x \, dx$

$$= \int (1-\cos^2 x) \sin x \, dx$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

$$-du = \sin x \, dx$$

$$= -\int (1-u^2) \, du$$

$$= -\left(u - \frac{u^3}{3}\right) + C$$

$$= -\left(\cos x - \frac{\cos^3 x}{3}\right) + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C$$

Example: Evaluate $\int \sin^4 x\, dx$ $= \int \left(\sin^2 x\right)^2 dx$

$$= \int \left[\frac{1}{2}\left(1-\cos 2x\right)\right]^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \underbrace{\cos^2 2x}_{\frac{1}{2}\left(1+\cos 4x\right)}\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left[x - \cancel{2}\,\frac{1}{\cancel{2}}\sin 2x + \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{4}\sin 4x\right] + C$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Example: Evaluate $\int \sin^5 x\, dx$ $= \int \sin^4 x \sin x\, dx$

$$(\sin^2 x)^2$$

$$(1 - \cos^2 x)^2$$

$$= \int (1-\cos^2 x)^2 \sin x\, dx$$

$$u = \cos x$$

$$du = -\sin x\, dx$$

$$-du = \sin x\, dx$$

$$= -\int (1-u^2)^2\, du = -\int (1-2u^2+u^4)\, du$$

$$= -u + \frac{2u^3}{3} - \frac{u^5}{5} + C$$

$$= -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + C$$

Example: Evaluate $\int \sin^5 x \cos^2 x \, dx$ $= \int \sin^4 x \cos^2 x \sin x \, dx$

$$= \int (1-\cos^2 x)^2 \cos^2 x \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1-u^2)^2 u^2 \, du$$

$$= -\int (u^2 - 2u^4 + u^6) \, du$$

$$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C$$

$$= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$$

In general, how to evaluate $\int \sin^m x \cos^n x \, dx$? $\equiv$

Case 1: n is odd, i.e. $n = 2k + 1$

$$\equiv \int \sin^m x \cos^{2k} x \cos x \, dx$$

$$= \int \sin^m x \left(\cos^2 x \right)^k \cos x \, dx$$

$$= \int \sin^m x \left(1 - \sin^2 x \right)^k \cos x \, dx$$

$$u = \sin x$$
$$du = \cos x \, dx$$

$$= \int u^m \left(1 - u^2 \right)^k du$$

Case 2: m is odd, i.e. $m = 2k + 1$

$$\int \sin^m x \cos^n x \, dx = \int \sin^{2k} x \cos^n x \, \sin x \, dx$$

$$= \int (1-\cos^2 x)^k \cos^n x \sin x \, dx$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

$$= -\int (1-u^2)^k u^n \, du$$

Case 3: both m and n are even

Use

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x \Rightarrow \sin^2 x \cos^2 x = \left[\frac{1}{2}\sin 2x\right]^2$$

Example: Evaluate $\int_0^{\pi} \sin^2 x \cos^4 x \, dx$ $= \int_0^{\pi} \underbrace{\sin^2 x \cos^2 x}\, \cos^2 x \, dx$

$$= \int_0^{\pi} \left(\frac{1}{2}\sin 2x\right)^2 \frac{1}{2}(1+\cos 2x)\, dx$$

$$= \frac{1}{8} \int_0^{\pi} (\sin 2x)^2 (1+\cos 2x)\, dx$$

$$= \frac{1}{8} \int_0^{\pi} (\sin 2x)^2 dx + \frac{1}{8} \int_0^{\pi} (\sin 2x)^2 \cos 2x \, dx$$

$$= \frac{1}{8}\int_0^{\pi} \sin^2 2x\,dx + \frac{1}{8}\int_0^{\pi} \sin^2 2x \cos 2x\,dx$$

$$\frac{1}{2}(1 - \cos 4x)$$

$u = \sin 2x$

$du = 2\cos 2x\,dx$

$\frac{1}{2}du = \cos 2x\,dx$

$$= \frac{1}{16}\int_0^{\pi} (1 - \cos 4x)\,dx + \frac{1}{8}\cdot\frac{1}{2}\int_{\sin(2\cdot 0)}^{\sin(2\pi)} u^2\,du$$

$$= \left[\frac{1}{16}x - \frac{1}{16}\frac{\sin 4x}{4}\right]_0^{\pi} + \frac{1}{16}\cdot\frac{u^3}{3}\Big|_0^0$$

$$= \frac{1}{16}\pi - \frac{1}{16}\cdot\frac{\sin 4\pi}{4} - 0 + 0$$

$$= \frac{\pi}{16}$$

Recall:

$$\boxed{\int \tan x \, dx = \ln|\sec x| + C}$$

Let's prove that

$$\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$$

<u>Proof</u>:

$$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{u} \, du$$

$$u = \sec x + \tan x$$

$$du = (\sec x \tan x + \sec^2 x) \, dx$$

$$= \ln|u| + c = \ln|\sec x + \tan x| + c$$

<u>Example</u>: Find $\int \sec^3 x\, dx$ $= \int \sec x \cdot \sec^2 x\, dx$

$\sec^2 x\, dx = dv$

$u = \sec x$

$du = \sec x \tan x\, dx$ $\qquad v = \tan x$

$= uv - \int v\, du = \sec x \tan x - \int \sec x \tan^2 x\, dx$

$\tan^2 x = \sec^2 x - 1$

$= \sec x \tan x - \int (\sec^3 x - \sec x)\, dx$

$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$

$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$

Example: Calculate $\int_0^{\pi/4} \tan^4 x \, dx$ $= \int_0^{\pi/4} \tan^2 x \tan^2 x \, dx$

$\tan^2 x = \sec^2 x - 1$

$= \int_0^{\pi/4} \tan^2 x (\sec^2 x - 1) \, dx = \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$= \int_0^1 u^2 \, du - \int_0^{\pi/4} \sec^2 x \, dx + \int_0^{\pi/4} 1 \, dx$

$= \frac{u^3}{3}\Big|_0^1 - \tan x \Big|_0^{\pi/4} + x \Big|_0^{\pi/4}$

$= \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{\pi}{4} - \frac{2}{3}$

Example: Evaluate $\int \tan^4 x \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$= \int u^4 \, du = \frac{u^5}{5} + C$

$= \frac{\tan^5 x}{5} + C$

In general, how to evaluate $\int \tan^m x \sec^n x \, dx$?

<u>Case 1</u>: n is even, i.e. $n = 2k, k \geq 2$

$$= \int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \sec^{2k-2} x \sec^2 x \, dx$$

$$\sec^{2k-2} x = (\sec^2 x)^{k-1}$$

$$= \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x \, dx$$

$$u = \tan x$$

$$du = \sec^2 x \, dx$$

$$= \int u^m (1+u^2)^{k-1} \, du$$

Case 2: m is odd, i.e. $m = 2k + 1$

$$\int \tan^{2k+1} x \sec^n x \, dx = \int \underbrace{\tan^{2k} x}_{(\tan^2 x)^k} \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int (u^2 - 1)^k u^{n-1} \, du$$

In other case, the solution might not be that clear.

Example: $\int \tan^3 x \sec^5 x \, dx$

$$= \int \tan^2 x \sec^4 x \quad \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1) \sec^4 x \sec x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int (u^2 - 1) u^4 \, du = \int (u^6 - u^4) \, du$$

$$= \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

Sometimes the following identities may be useful:

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

<u>Example</u>: Evaluate $\int \sin 4x \sin 3x \, dx$

$$= \frac{1}{2} \int [\cos(4x-3x) - \cos(4x+3x)] \, dx$$

$$= \frac{1}{2} \int [\cos x - \cos 7x] \, dx$$

$$= \frac{1}{2} \left[\sin x - \frac{1}{7}\sin 7x\right] + C$$

Trigonometric Substitution

Let's find

$$\int x\sqrt{a^2 - x^2}\,dx\,,\;\; a > 0$$

$u = a^2 - x^2$

$du = -2x\,dx$

$x\,dx = -\frac{1}{2}\,du$

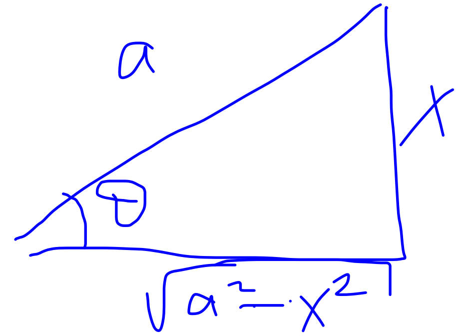
$= -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} + C$

$= -\frac{1}{3}\left(a^2 - x^2\right)^{3/2} + C$

What about $\int \sqrt{a^2 - x^2}dx$?

$$\sin\theta = \frac{x}{a}$$

$$\theta = \sin^{-1}\frac{x}{a}$$



Let

$$x = a\sin\theta$$

This kind of substitution is called **inverse substitution**, provided that it defines a one-to-one function, i.e. $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$= \int \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta = a^2 \int \cos^2\theta\, d\theta$$

$$dx = a\cos\theta\, d\theta \qquad \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta}$$

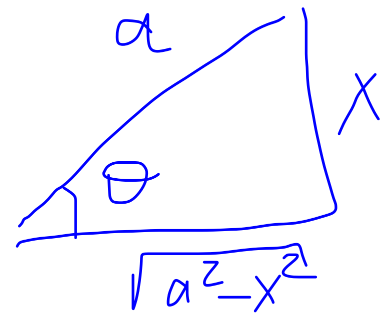
$$= a^2\left[\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right] + C$$

$$= a^2\left[\frac{1}{2}\cdot\frac{x}{a}\,\frac{\sqrt{a^2 - x^2}}{a} + \frac{1}{2}\sin^{-1}\frac{x}{a}\right] + C$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

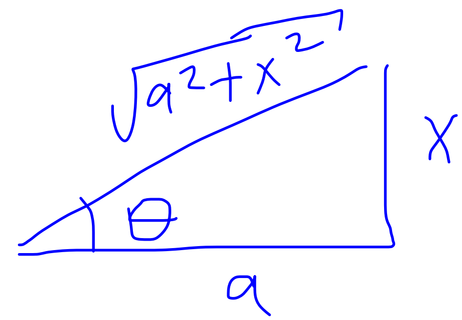
Trigonometric Substitutions:

- For $\sqrt{a^2 - x^2}$ define $x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



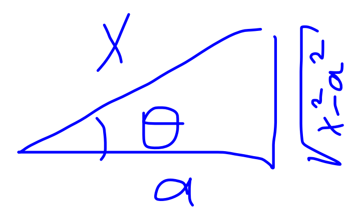
$$1 - \sin^2\theta = \cos^2\theta$$

- For $\sqrt{a^2 + x^2}$ define $x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



$$1 + \tan^2\theta = \sec^2\theta$$

- For $\sqrt{x^2 - a^2}$ define $x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$



$$\sec^2\theta - 1 = \tan^2\theta$$

<u>Example</u>: Calculate $\int \sqrt{a^2+x^2}\,dx$ $= \int \sqrt{a^2 + a^2\tan^2\theta}\; a\sec^2\theta\, d\theta$

$x = a\tan\theta$

$a^2(1+\tan^2\theta)$

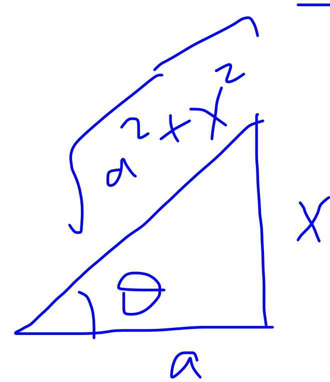
$dx = a\sec^2\theta\, d\theta$

$a^2\sec^2\theta$

$$= a^2 \int \sec^3\theta\, d\theta$$

$$= a^2 \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| \right] + C$$

was calculated earlier



$$= a^2 \left[\frac{1}{2} \frac{\sqrt{a^2+x^2}}{a} \cdot \frac{x}{a} + \frac{1}{2} \ln\left| \frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a} \right| \right] + C$$

$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|\sqrt{a^2+x^2} + x\right| + C_0$$

$\sec\theta = \dfrac{\sqrt{a^2+x^2}}{a}$

Example: Find $\int \frac{x^5}{\sqrt{x^2+2}}\,dx$ $= \int \frac{4\sqrt{2}\tan^5\theta}{\sqrt{2\tan^2\theta+2}}\sqrt{2}\sec^2\theta\,d\theta$

$x^2+(\sqrt{2})^2$

$x=\sqrt{2}\tan\theta$ $\quad x^5=(\sqrt{2}\tan\theta)^5=4\sqrt{2}\tan^5\theta$

$dx=\sqrt{2}\sec^2\theta\,d\theta$

$$= \int \frac{4\sqrt{2}\tan^5\theta\,\cancel{\sqrt{2}}\sec^{\cancel{2}}\theta}{\cancel{\sqrt{2}}\,\cancel{\sec\theta}}\,d\theta$$

$$= 4\sqrt{2}\int \tan^5\theta\sec\theta\,d\theta$$

$$= 4\sqrt{2}\int \tan^4\theta\sec\theta\tan\theta\,d\theta$$

$(\tan^2\theta)^2=(\sec^2\theta-1)^2$

$$= 4\sqrt{2}\int (\sec^2\theta - 1)^2 \sec\theta\tan\theta\, d\theta$$

$$u = \sec\theta$$

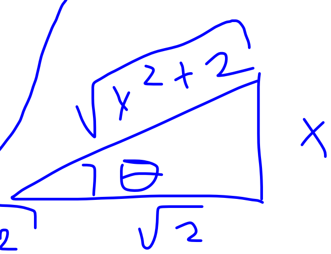
$$du = \sec\theta\tan\theta\, d\theta$$

$$= 4\sqrt{2}\int (u^2-1)^2\, du = 4\sqrt{2}\int (u^4 - 2u^2 + 1)\, du$$

$$= 4\sqrt{2}\left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right) + C$$

$$= 4\sqrt{2}\left[\frac{\sec^5\theta}{5} - \frac{2}{3}\sec^3\theta + \sec\theta + C\right.$$

$$= 4\sqrt{2}\left[\frac{1}{5}\left(\frac{x^2+2}{2}\right)^{5/2} - \frac{2}{3}\left(\frac{x^2+2}{2}\right)^{3/2} + \left(\frac{x^2+2}{2}\right)^{1/2}\right] + C$$



$$\sec\theta = \frac{\sqrt{x^2+2}}{\sqrt{2}}$$

<u>Example</u>: Evaluate $\int \frac{dx}{x^2\sqrt{x^2-9}}$

$\sqrt{x^2-3^2}$

$$= \int \frac{\cancel{3}\,\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{9\sec^{\cancel{2}}\theta \cdot \underbrace{\cancel{3}\,\cancel{\tan\theta}}_{\sqrt{9\sec^2\theta-9}}}$$

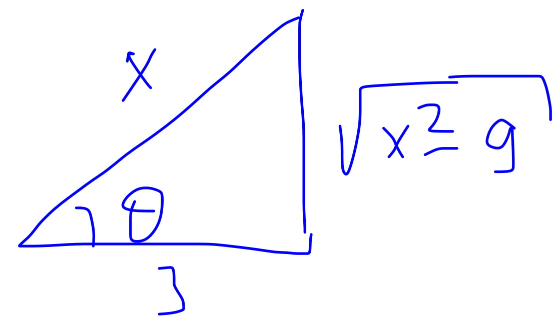
$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$= \frac{1}{9}\int \frac{d\theta}{\sec\theta} = \frac{1}{9}\int \cos\theta\,d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \frac{1}{9}\cdot\frac{\sqrt{x^2-9}}{x} + C$$



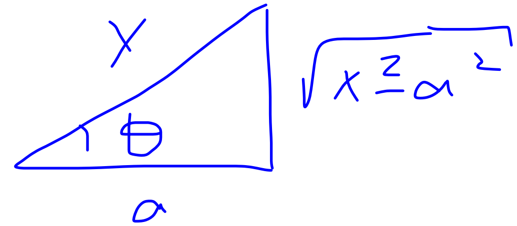
$$\sin\theta = \frac{\sqrt{x^2-9}}{x}$$

Example: Find $\int \sqrt{x^2 - a^2}\,dx$ $= \int \sqrt{a^2 \sec^2\theta - a^2}\, \sec\theta \tan\theta\, d\theta$

$x = a\sec\theta$

$dx = a \sec\theta \tan\theta\, d\theta$

$\tan\theta = \dfrac{\sqrt{x^2 - a^2}}{a}$



$$= a^2 \int \sec\theta \tan^2\theta\, d\theta = a^2 \int \sec\theta(\sec^2\theta - 1)\, d\theta$$

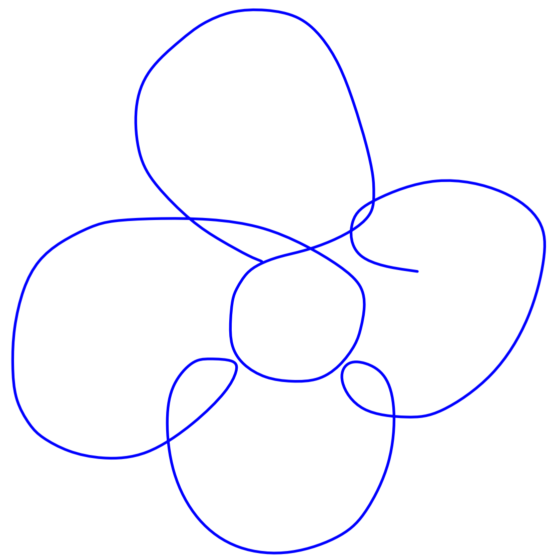
$$= a^2 \int \sec^3\theta\, d\theta - a^2 \int \sec\theta\, d\theta$$

$$= a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]$$

$$- a^2 \ln|\sec\theta + \tan\theta| + C$$

$$= \frac{a^2}{2}\left[\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right] + C$$

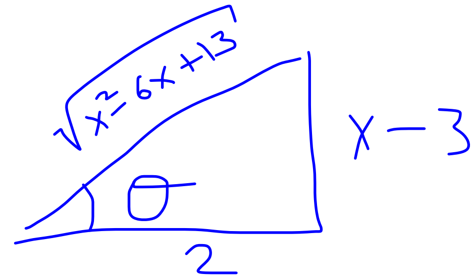
$$= \frac{a^2}{2}\left[\frac{x}{a}\,\frac{\sqrt{x^2 - a^2}}{a} - \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right|\right] + C$$



Example: Evaluate $\int \frac{dx}{\sqrt{x^2-6x+13}}$ $= \int \frac{dx}{\sqrt{(x-3)^2+2^2}}$

$$x^2-6x+13 = x^2-6x+9+4 = (x-3)^2+2^2$$

$$x-3 = 2\tan\theta \qquad \sec\theta = \frac{\sqrt{x^2-6x+13}}{2}$$



$$dx = 2\sec^2\theta\, d\theta$$

$$= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{2^2\tan^2\theta+2^2}} = \int \frac{\sec^{\cancel{2}}\theta}{\cancel{\sec\theta}}\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta|+C$$

$$= \ln\left|\frac{\sqrt{x^2-6x+13}}{2} + \frac{x-3}{2}\right| + C$$

<u>Example</u>: Calculate $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du$

$u = \sin x$

$du = \cos x\,dx$

$u = \tan\theta \Rightarrow \theta = \tan^{-1} u$

$du = \sec^2\theta\,d\theta$

$$= \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} = \int_0^{\pi/4} \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 - \tan 0|$$

$$= \ln|\sqrt{2}+1| - \ln|1-0| = \ln(\sqrt{2}+1)$$