Lecture 8 (Integrals continued)

Indefinite Integrals

Recall: Part I of FTC says that if f is continuous, then $\int_a^x f(t)dt$ is an antiderivative of f. Part II tells us that $\int_a^b f(x)dx = F(b) - F(a)$, where F is an antiderivative. From now on we shall use the following notation for an antiderivative:

$$\int f(x)dx = F(x)$$

and call it an **indefinite integral**.

Example:

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int \cos x \, dx = \sin x + C$$

Caution: There is a difference between a definite integral $\int_a^b f(x)dx$ which is a number, and an indefinite integral $\int f(x)dx$ which is a function (or a family of functions).

$$\int_a^b f(x)dx = \int f(x)dx\Big|_a^b$$

Properties of indefinite integrals:

- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
- $\int cf(x)dx = c\int f(x)dx$

Table of Indefinite Integrals

$$\int k\,dx = kx + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \sec^2 x\, dx = \tan x + C$$

$$\int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x \tan x\, dx = \sec x + C$$

$$\int \csc x \cot x\, dx = -\csc x + C$$

$$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

Examples: Find the indefinite integrals

$$\int \frac{1}{x^3}dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$= -\frac{1}{2x^2} + C$$

$$\int \frac{x^2-\sqrt{x}+1}{x}dx = \int \left[x - x^{-1/2} + \frac{1}{x}\right] dx$$

$$= \int x\,dx - \int x^{-1/2}\,dx + \int \frac{1}{x}\,dx$$

$$= \frac{x^2}{2} - 2x^{1/2} + \ln|x| + C$$

$$\int (3\sec^2 x - 5e^x)dx = 3\int \sec^2 x\,dx - 5\int e^x dx$$

$$= 3\tan x - 5e^x + C$$

<u>Examples</u>: Evaluate

$$\int_1^2 \frac{(x-1)^2}{x}\,dx = \int_1^2 \frac{x^2-2x+1}{x}\,dx = \int_1^2 \left(x-2+\frac{1}{x}\right)dx$$

$$= \left[\frac{x^2}{2} - 2x + \ln|x|\right]_1^2$$

$$= \frac{2^2}{2} - 2\cdot 2 + \ln 2 - \left(\frac{1^2}{2} - 2\cdot 1 + \ln 1\right)$$

$$= -2 + \ln 2 + \frac{3}{2}$$

$$= -\frac{1}{2} + \ln 2$$

$\int_0^2 |2x-1|dx =$

$$|2x-1| = \begin{cases} 2x-1, & x \geq \frac{1}{2} \\ -(2x-1), & x \leq \frac{1}{2} \end{cases}$$

$$= \int_0^{1/2} (1-2x)\,dx + \int_{1/2}^{2} (2x-1)\,dx$$

$$= \left[x - 2 \cdot \frac{x^2}{2}\right]_0^{1/2} + \left[2 \cdot \frac{x^2}{2} - x\right]_{1/2}^{2}$$

$$= \left[\frac{1}{2} - \frac{1}{4} - 0\right] + \left[(4-2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2 = \frac{5}{2}$$

$$\int_0^{\pi/3} \frac{\sin x + \sin x \tan^2 x}{\sec^2 x}\, dx =$$

$$= \int_0^{\pi/3} \frac{\sin x\,(1 + \tan^2 x)}{\sec^2 x}\, dx$$

$$= \int_0^{\pi/3} \frac{\sin x \cdot \cancel{\sec^2 x}}{\cancel{\sec^2 x}}\, dx$$

$$= \int_0^{\pi/3} \sin x \, dx = -\cos x \Big|_0^{\pi/3}$$

$$= -\left[\cos \frac{\pi}{3} - \cos 0\right]$$

$$= -\left[\frac{1}{2} - 1\right] = \frac{1}{2}$$

Substitution Rule

Consider the following integral:

$$\int \frac{2x}{x^2+3}\,dx = \int \underbrace{\frac{1}{x^2+3}}_{\frac{1}{u}} \cdot \underbrace{2x\,dx}_{du}$$

$u = x^2 + 3$

$u' = 2x$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|x^2+3| + C$$

$$= \ln(x^2+3) + C$$

In general, this method will work if we have an integral of the form

$$\int f\big(g(x)\big)g'(x)dx \quad = \int f(u)\,du$$

If F is an antiderivative of f, then $F' = f$. Thus,

$$\int F'\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + C \quad \text{(by the Chain Rule)}$$

If we make a substitution $u = g(x)$, then $du = g'(x)dx$, and we get that

$F(g(x))$ is antiderivative of $f(g(x))g'(x)$

why? $[F(g(x))]' = F'(g(x)) \cdot g'(x)$

$= f(g(x))\,g'(x)$

So $\int f(g(x))g'(x)dx = F(g(x)) + c$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval on which f is continuous, then

$$\int f\big(g(x)\big)g'(x)dx = \int f(u)du$$

$u = g(x), \quad du = g'(x)\,dx$

Example: Evaluate

$$\int \sqrt{x+1}dx = \int \sqrt{u}\,du = \frac{u^{3/2}}{3/2} + C$$

$u = x+1$

$du = 1 \cdot dx$

$$= \frac{2}{3}(x+1)^{3/2} + C$$

$$\int e^{-2x}\,dx = \int e^u\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int e^u\,du$$

$u = -2x$

$du = -2\,dx$

$-\frac{1}{2}du = dx$

$$= -\frac{1}{2}e^u + C$$

$$= -\frac{1}{2}e^{-2x} + C$$

$$\int x^2\sin(x^3-1)\,dx = \frac{1}{3}\int \sin u\,du$$

$u = x^3 - 1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$= -\frac{1}{3}\cos u + C$$

$$= -\frac{1}{3}\cos(x^3-1) + C$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

$$-du = \sin x \, dx$$

$$= -\ln |u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln \frac{1}{|\cos x|} + C$$

$$= \ln|\sec x| + C$$

$$\int xe^{x^2}dx = \frac{1}{2}\int e^u\,du$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}e^u + c$$

$$= \frac{1}{2}e^{x^2} + c$$

$$\int x\sqrt{x+2}\,dx = \int (u-2)\sqrt{u}\;du$$

$u = x+2 \Rightarrow x = u-2$

$du = dx$

$$= \int \left[u\sqrt{u} - 2\sqrt{u}\right] du = \int \left[u^{3/2} - 2u^{1/2}\right] du$$

$$= \frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$$

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

$$u = 1 + e^x \qquad = \ln|1+e^x| + C$$

$$du = e^x\,dx \qquad = \ln(1+e^x) + C$$

Substitution Rule for Definite Integrals:

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$(*) \quad \int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Proof: Let F be antiderivative of f

$\Longrightarrow F(g(x))$ is antiderivative of $f(g(x))g'(x)$

$$\int_a^b f(g(x))g'(x)dx \underset{\text{FTC, II}}{=} F(g(b)) - F(g(a))$$

$$\| $$

$$\int_{g(a)}^{g(b)} f(u)du \underset{\text{FTC, II}}{=} F(g(b)) - F(g(a))$$

$\Longrightarrow (*)$ ∎

Example: Evaluate

$$\int_0^3 \sqrt{5x+2}\,dx =$$

$u = 5x+2$

$du = 5dx$

$\frac{1}{5}du = dx$

$$\frac{1}{5}\int_{5\cdot 0+2}^{5\cdot 3+2} \sqrt{u}\,du = \frac{1}{5}\int_2^{17} \sqrt{u}\,du$$

$$= \frac{1}{5}\left.\frac{u^{3/2}}{3/2}\right|_2^{17}$$

$$= \frac{2}{15}\left[17^{3/2} - 2^{3/2}\right]$$

$$= 8.96$$

$$\int_1^e \frac{\ln x}{x}\,dx = \int_{\ln 1}^{\ln e} u\,du = \int_0^1 u\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_{-0^2}^{-1^2} e^u\,du = -\frac{1}{2}\int_0^{-1} e^u\,du$$

$$u = -x^2$$

$$du = -2x\,dx$$

$$-\frac{1}{2}\,du = x\,dx$$

$$= \frac{1}{2}\int_{-1}^{0} e^u\,du$$

$$= \frac{1}{2}\, e^u \Big|_{-1}^{0}$$

$$= \frac{1}{2}\left(e^0 - e^{-1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{e}\right)$$

$$\int_0^{\pi/2} \cos x \sin(\sin x)\, dx = \int_{\sin 0}^{\sin\frac{\pi}{2}} \sin u \, du$$

$u = \sin x$

$du = \cos x \, dx$

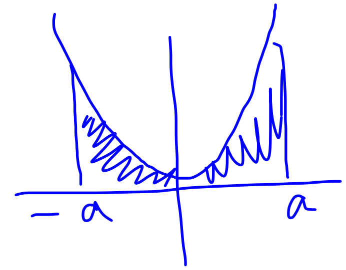
$$= \int_0^1 \sin u \, du$$

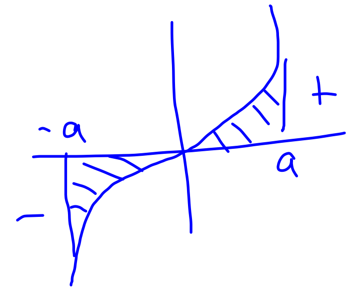
$$= -\cos u \Big|_0^1$$

$$= -(\cos 1 - \cos 0)$$

$$= 1 - \cos 1$$

Symmetry





Suppose f is continuous on $[-a, a]$.

- If f is even, i.e. $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
- If f is odd, i.e. $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)dx = 0$

Proof:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$u = -x$

$du = -dx$

$$= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

Proof (continued):

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

If $f(x)$ is even, $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

If $f(x)$ is odd, $f(-x) = -f(x)$, then

$$\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0$$

$\blacksquare$

Example: Evaluate

$$\int_{-\pi/3}^{\pi/3} x^4 \cos x \sin^3 x \, dx = 0$$

$$f(-x) = (-x)^4 \cos(-x) \sin^3(-x)$$

$$= x^4 \cos x \left(-\sin^3 x\right)$$

$$= -x^4 \cos x \sin^3 x = -f(x)$$

$f(x)$ is odd