Lecture 7 (Integrals)

Antiderivatives (section 4.9)

Very often we need to solve a reverse problem: we need to find a function, $F(x)$, whose derivative, $f(x)$, is known.

Definition: A function F is called an **antiderivative** of f on (a, b), if

$$F'(x) = f(x), \qquad x \in (a, b)$$

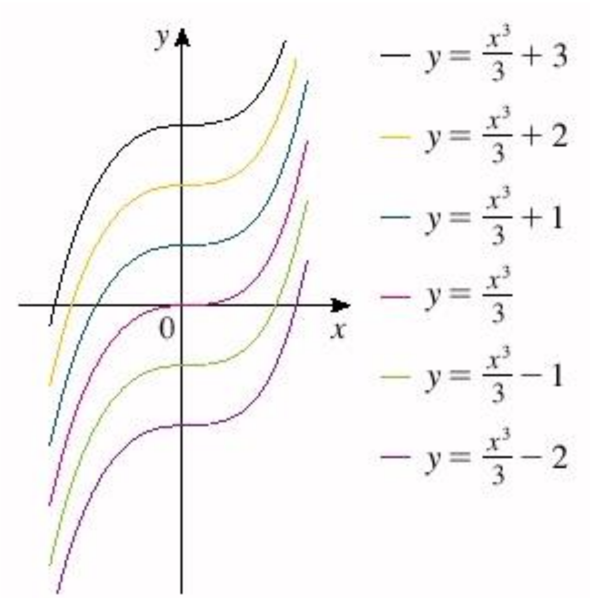
Example: $f(x) = x^2$

$$\frac{1}{3}(x^3)' = \frac{1}{3} 3x^2 = x^2$$

$$F(x) = \frac{1}{3}x^3$$

Check: $F'(x) = \left(\frac{1}{3}x^3\right)' = \frac{1}{3} \cdot 3x^2 = x^2$

$$G'(x) = \left(\frac{1}{3}x^3 + 5\right)' = x^2$$



$\rightarrow \left(\frac{x^3}{3}+2\right)' \stackrel{?}{=} x^2$

Theorem: If F is an antiderivative of f on (a, b), then the most general antiderivative of f on (a, b) is

$$F(x) + C$$

where C is a constant

Example: $f(x) = x^n$

$$\frac{1}{n+1}\left(x^{n+1}\right)' = \frac{(n+1)x^n}{n+1} = x^n$$

$$F(x) = \frac{1}{n+1}x^{n+1} + C$$

Function	Antiderivative
x^n	$\frac{x^{n+1}}{n+1}+C$
$\frac{1}{x}$	$\ln\lvert x\rvert+C$
e^x	e^x+C
$\cos x$	$\sin x+C$
$\sin x$	$-\cos x+C$
$\sec^2 x$	$\tan x+C$
$\sec x\tan x$	$\sec x+C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x+C$
$\frac{1}{1+x^2}$	$\tan^{-1}x+C$

<u>Example</u>: $f(x) = 2e^x - \sin x + 1$. Find the antiderivative, $F(x)$.

$$F(x) = 2e^x + \cos x + x + C$$

Check:

$$F'(x) = 2e^x - \sin x + 1 = f(x)$$

Example: Given $f''(x) = 2 + \cos x$, $f(0) = -1$, $f\left(\frac{\pi}{2}\right) = 0$. Find f.

antiderivative of $f''(x)$ $\leftarrow$ $f'(x) = 2x + \sin x + C$

Check: $f''(x) = 2 + \cos x$

antiderivative of $f'(x)$ $\leftarrow$ $f(x) = 2 \cdot \frac{x^2}{2} - \cos x + Cx + D$

$= x^2 - \cos x + Cx + D$

Check: $f'(x) = 2x + \sin x + C$

$-1 = f(0) = 0^2 - 1 + C \cdot 0 + D \Rightarrow D = 0$

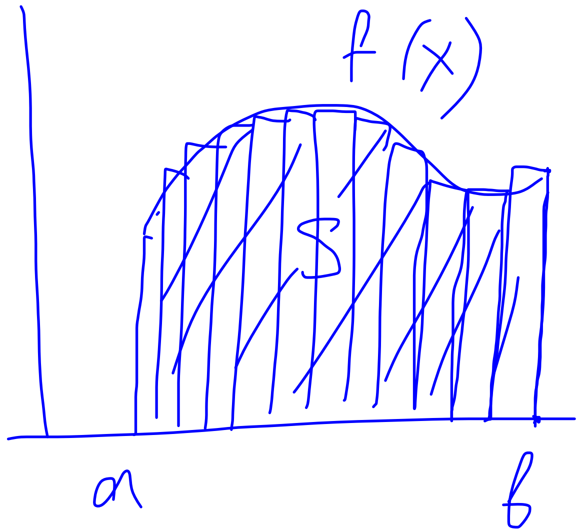
$0 = f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 - 0 + C \cdot \frac{\pi}{2} + 0 \Rightarrow C = -\frac{\pi}{2}$

$\boxed{f(x) = x^2 + \cos x - \frac{\pi}{2}x}$

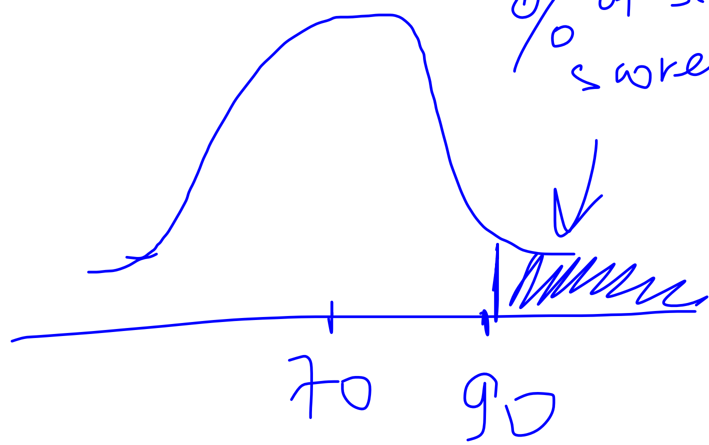
Area Problem

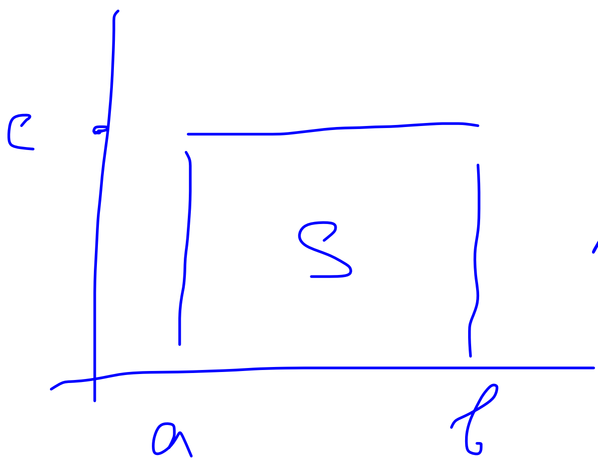
Find the area of the region under the curve $y = f(x)$ from a to b.

$A_S = ?$

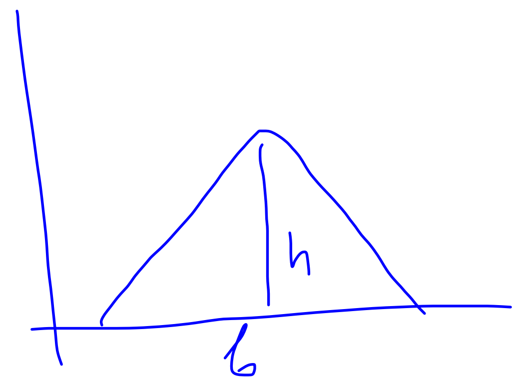


% of students scored ≥ 90



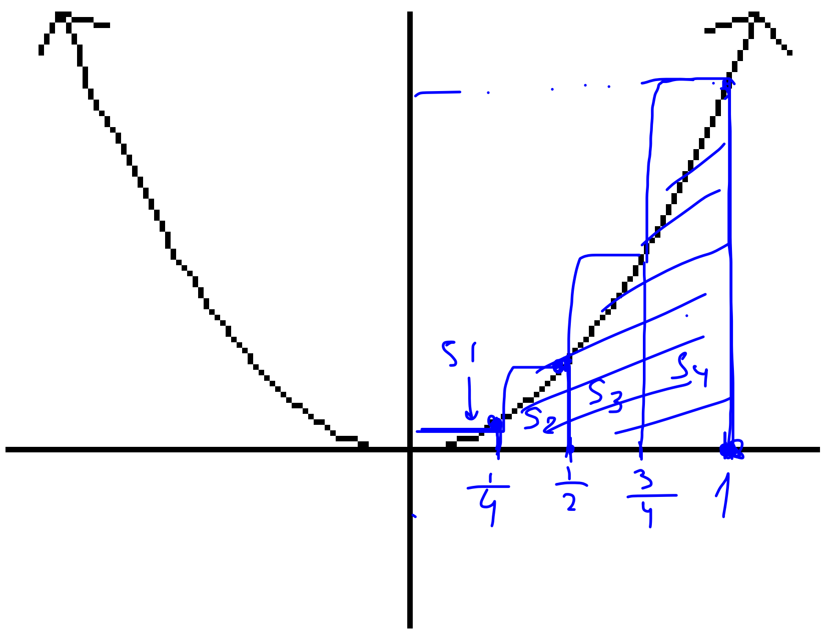


$A_S = c(b-a)$



$A_S = \frac{1}{2}\, b \cdot h$

<u>Example</u>: Estimate the area under $y = x^2$ on $(0, 1)$.



$0 < A_S < 1$

$A_S = A_{S_1} + A_{S_2} + A_{S_3} + A_{S_4}$

$A_{S_1} \approx$ area of rectangle

$= \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2$

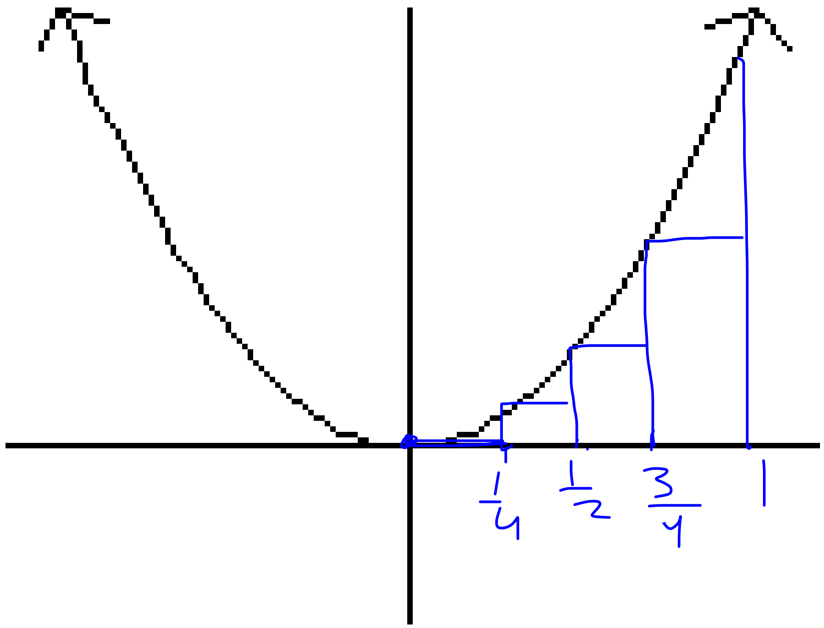
$A_{S_2} \approx \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2$

$A_S \approx \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2$

$= 0.469 = R_4 \rightarrow$ # of intervals

right end point

$A_S < 0.469$



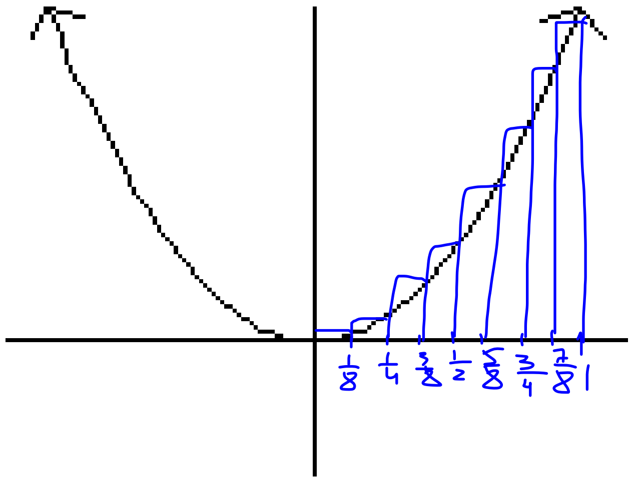
$$L_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2$$

left end point

$$= 0.219$$

$$A_S > 0.219$$

$$0.219 < A_S < 0.469$$



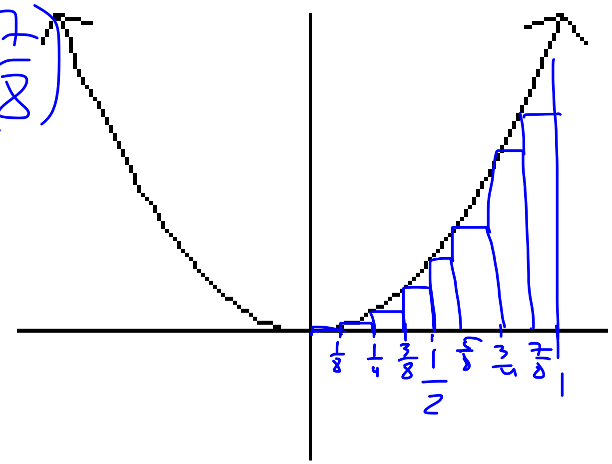
$$R_8 = \frac{1}{8}\cdot\left(\frac{1}{8}\right)^2 + \frac{1}{8}\left(\frac{1}{4}\right)^2 + \dots + \frac{1}{8}\cdot 1^2$$

$$= 0.398$$

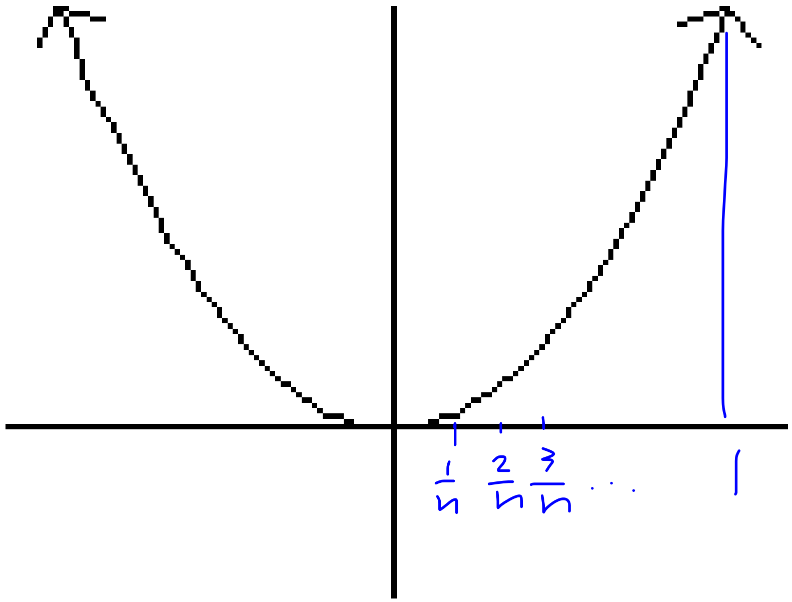
$$A_S < 0.398$$

$$L_8 = \frac{1}{8}\cdot\left(\frac{1}{8}\right)^2 + \frac{1}{8}\cdot\left(\frac{1}{4}\right)^2 + \dots + \frac{1}{8}\cdot\left(\frac{7}{8}\right)^2$$

$$= 0.273 < A_S$$



$$0.273 < A_S < 0.398$$



$$A_S \approx \frac{0.3328 + 0.3338}{2}$$

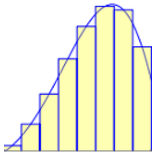
$$\approx 0.333$$

n	L_n	R_n
10	0.285	0.385
30	0.317	0.350
100	0.328	0.338
1000	0.3328	0.3338

$A_S =$ limit [sum of areas of rectangles]

as the width of rectangle $\to 0$

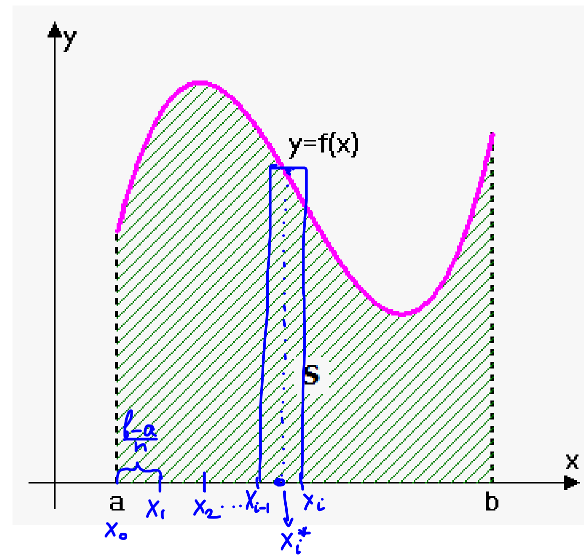
$n \to \infty$



Riemann Sums

Our goal is to find a formula for the area which is bounded by the graph of the continuous function f defined on $[a, b]$.

To achieve that, we divide the interval $[a, b]$ into n equal subintervals of the width $\frac{b-a}{n}$. Then we choose points $x_i^* \in [x_{i-1}, x_i]$, called **sample points**.



We denote by R_i the area of the rectangle with height $f(x_i^*)$ and width $\boxed{\Delta x = \frac{b-a}{n}}$, i.e.

$$R_i = f(x_i^*) \cdot \Delta x$$

height width

Definition: The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

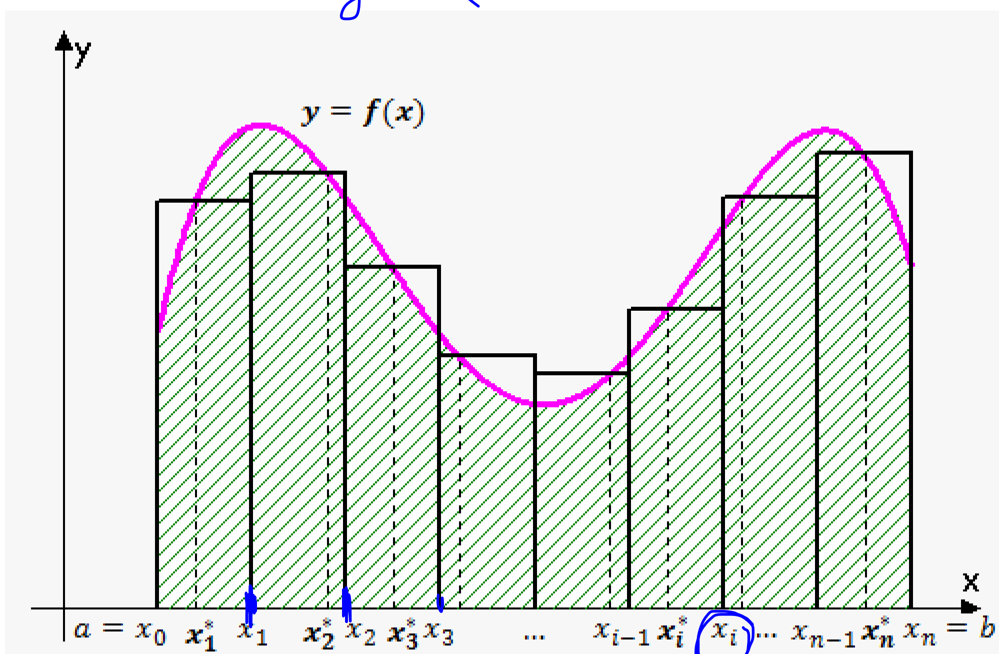
$$A = \lim_{n\to\infty}[R_1 + \cdots + R_n] = \lim_{n\to\infty}[\,f(x_1^*)\cdot \underbrace{\Delta x}_{=\frac{b-a}{n}} + \cdots + f(x_n^*)\cdot \Delta x]$$

In **sigma notation**: $A = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n}}_{\text{sigma}} f(x_i^*)\cdot \Delta x$

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n$$

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$$

$$\sum_{i=1}^{3} i = 1 + 2 + 3$$



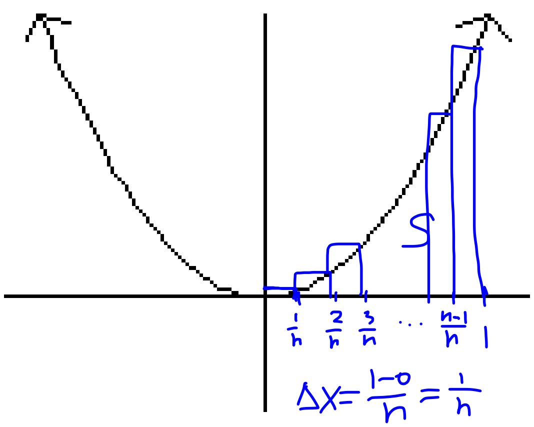
<u>Definition</u>: The sum

$$\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

is called a **Riemann sum**.

Back to our

<u>Example</u>: Estimate the area under $y = x^2$ on $(0, 1)$



$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\, \Delta x$$

right end point

$x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, $x_3 = \frac{3}{n}$, ..., $x_n = 1$

$x_i = \frac{i}{n}$

$$A = \lim \sum f(x_i)\,\Delta x$$

$$x_i = \frac{i}{n}, \quad \Delta x = \frac{1}{n}, \quad f(x) = x^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \Big[\underbrace{\left(\frac{1}{n}\right)^2}_{f(x_i)} \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + \dots + \left(\frac{n}{n}\right)^2 \cdot \frac{1}{n} \Big]$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \left[1^2 + 2^2 + 3^2 + \dots + n^2 \right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

Useful formula:

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$A = \lim_{n\to\infty} \frac{1}{n^{\cancel{3}}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{1}{6} \lim \frac{1}{n^2} (n+1)(2n+1)$$

$$= \frac{1}{6} \lim_{n\to\infty} \frac{1}{n}(n+1) \cdot \frac{1}{n}(2n+1)$$

$$= \frac{1}{6} \lim_{n\to\infty} (1 + \underbrace{\tfrac{1}{n}}_{\to 0})(2 + \underbrace{\tfrac{1}{n}}_{\to 0}) = \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}$$

$$\int_a^b f(x)dx$$

Definite Integral

We tried to estimate the area of the region by subdividing the interval into many subintervals. Our goal is to get the exact area, not just an estimate (like we did in the previous example).

Definition: Let f be a function defined on $[a, b]$. We divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$, and let $a = x_0, x_1, x_2, \dots, x_{n-1}, x_n = b$ be the endpoints of these subinterval and we let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points of these subintervals, i.e. $x_i^* \in [x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. If it does exist, f is called **integrable** on $[a, b]$.

upper limit

integral sign

$$\int_a^b f(x)\,dx$$

we integrate w.r.t. x

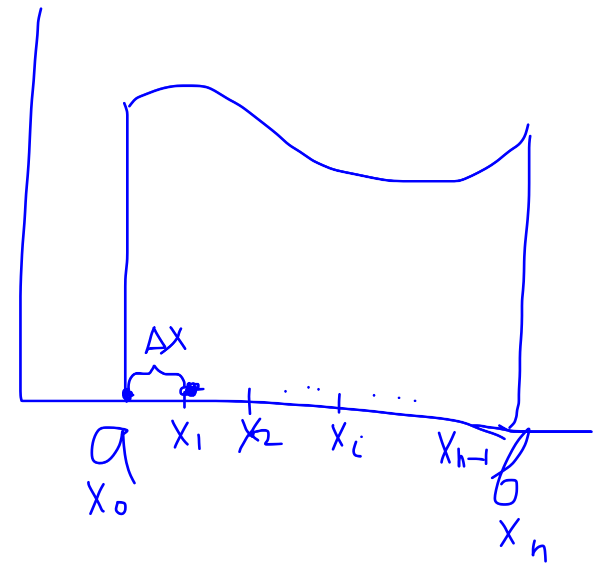
integrand

lower limit

Theorem: If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $\boxed{x_i = a + i\Delta x \text{ (right endpoints)}}$.



Example: Evaluate $\int_0^2 (2x - x^3)dx$

$f(x) = 2x - x^3, \quad \Delta x = \frac{2-0}{n} = \frac{2}{n}$

$x_i = a + i\Delta x = 0 + i \cdot \frac{2}{n} = \frac{2}{n} i$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$\vdots$

$x_i = a + i\Delta x$

$$\int_0^2 (2x - x^3)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[2 \cdot \frac{2}{n} i - \left(\frac{2}{n} i\right)^3\right] \frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{4}{n} i + \frac{8}{n^3} i^3\right] \cdot \frac{2}{n}$$

Rules for sigma notation:

- $\sum_{i=1}^{n} c = nc$

$$\sum_{i=1}^{n} c = \underbrace{c + c + \dots + c}_{n \text{ times}} = nc$$

- $\sum_{i=1}^{n} cx_i = c \sum_{i=1}^{n} x_i$

$$\sum_{i=1}^{n} cx_i = cx_1 + cx_2 + \dots + cx_n = c(x_1 + x_2 + \dots + x_n)$$
$$= c \sum_{i=1}^{n} x_i$$

- $\sum_{i=1}^{n} (x_i \pm y_i) = \sum_{i=1}^{n} x_i \pm \sum_{i=1}^{n} y_i$

$$\sum_{i=1}^{n} (x_i \pm y_i) = x_1 \pm y_1 + x_2 \pm y_2 + \dots + x_n \pm y_n$$
$$= (x_1 + x_2 + \dots + x_n) \pm (y_1 + y_2 + \dots + y_n)$$
$$= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$$

Useful formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = 1+2+3+\dots+n$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = 1^2+2^2+3^2+\dots+n^2$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = 1^3+2^3+3^3+\dots+n^3$$

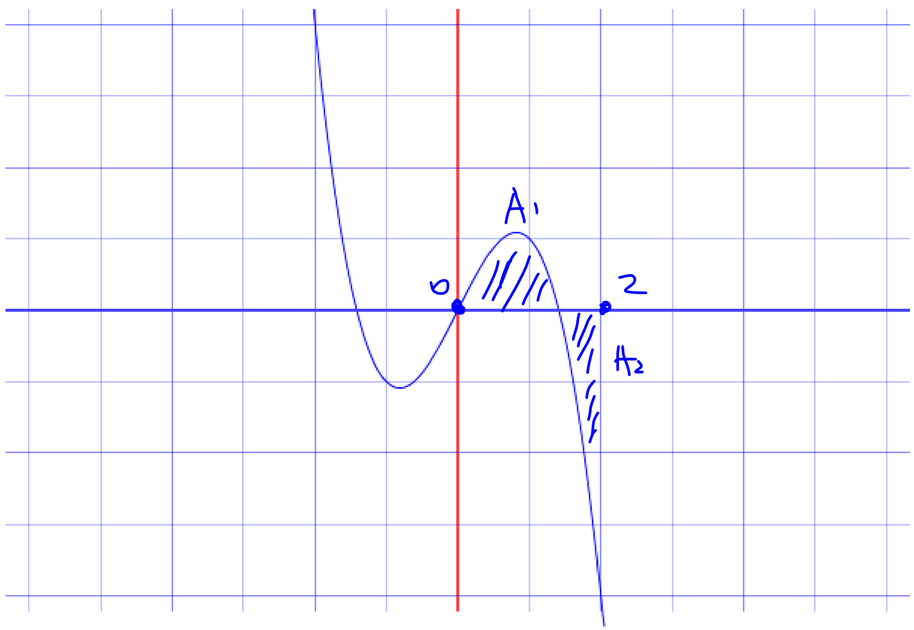
Back to $\int_0^2 (2x - x^3)dx$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{4}{n} i - \frac{8}{n^3} i^3\right]\frac{2}{n} = \lim \left[\frac{4}{n}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^3\right]\frac{2}{n}$$

$$= \lim_{n\to\infty} \left[\frac{4}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{2} - \frac{8}{n^{\cancel{3}}} \cdot \left(\frac{\cancel{n}(n+1)}{2}\right)^2\right] \cdot \frac{2}{n}$$

$$= \lim \left[\frac{4}{n}(n+1) - \frac{4}{n^2}(n+1)^2\right] = 4 \lim \left[\frac{n+1}{n} - \left(\frac{n+1}{n}\right)^2\right]$$

$$= 4 \lim_{n\to\infty} \left[1 + \frac{1}{n} - \left(1+\frac{1}{n}\right)^2\right] = 0$$



$$\int_0^2 (2x - x^3)\,dx$$

$$= A_1 - A_2$$

Properties of Definite Integral

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

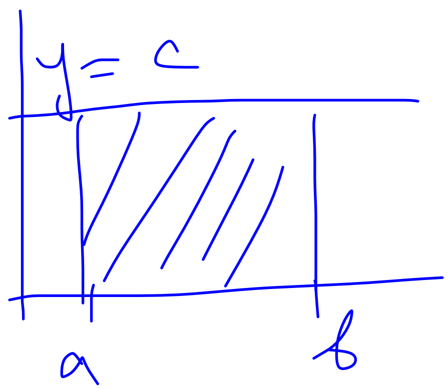
Proof:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\frac{b-a}{n} = -\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\frac{a-b}{n} = -\int_b^a f(x)\,dx$$

- $\int_a^a f(x)dx = 0$ since $\frac{a-a}{n} = 0$

- $\int_a^b c\,dx = c(b-a)$

Proof:



$A = c(b-a)$

$$\int_a^b c\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} c\,\frac{b-a}{n}$$

$$= \lim_{n\to\infty} c \sum_{i=1}^{n} \frac{b-a}{n}$$

$$= \lim c \cdot \frac{b-a}{\cancel{n}} \cdot \cancel{n}$$

$$= c(b-a)$$

- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

<u>Proof</u>:

$$\int_a^b [f(x)+g(x)]dx = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i)+g(x_i)]\Delta x$$

$$= \lim \sum f(x_i)\Delta x + \lim \sum g(x_i)\Delta x$$

$$= \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\Delta x = \frac{b-a}{n}$$

Example: Evaluate $\int_0^1 (1 - 2x^2)dx$

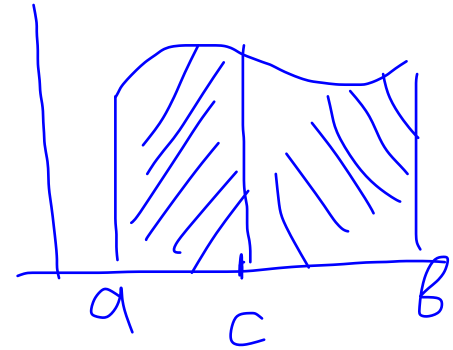
$$\int_0^1 (1 - 2x^2)\,dx = \int_0^1 1\,dx - 2\int_0^1 x^2\,dx$$

$$= 1 \cdot (1-0) - 2 \cdot \frac{1}{3}$$

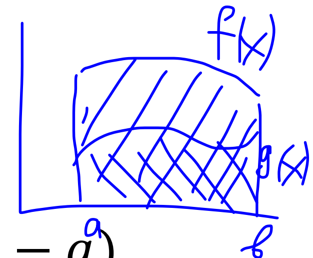
calculated it earlier

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

More properties...

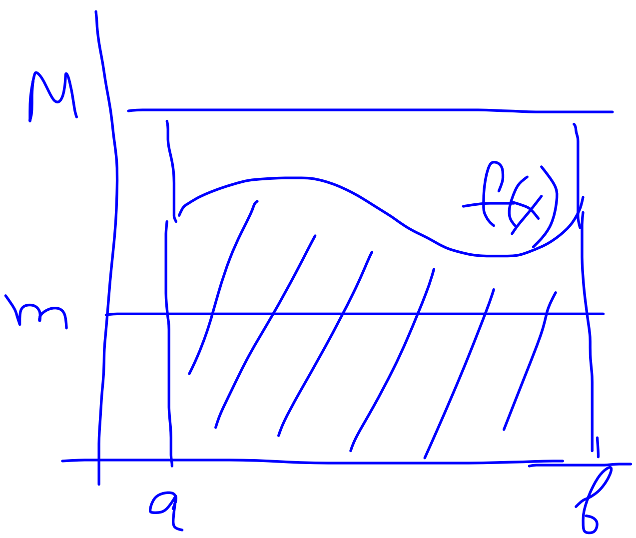


- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx, \quad a < c < b$



- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Proof:



area under $y=m$ $\leq \int_a^b f(x)\,dx \leq$ area under $y=M$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

<u>Example</u>: Use the last property to estimate $\int_{-1}^{1} \sqrt{1+x^2}\,dx$.

$$f(x) = \sqrt{1+x^2}, \quad [-1, 1]$$

$$-1 \leq x \leq 1$$

$$0 \leq x^2 \leq 1$$

$$1 \leq 1+x^2 \leq 2$$

$$m = \sqrt{1} \leq \sqrt{1+x^2} \leq \sqrt{2} = M$$

$$\sqrt{1}(1-(-1)) \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq \sqrt{2}\,(1-(-1))$$

$$2 \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$$

Fundamental Theorem of Calculus (FTC)

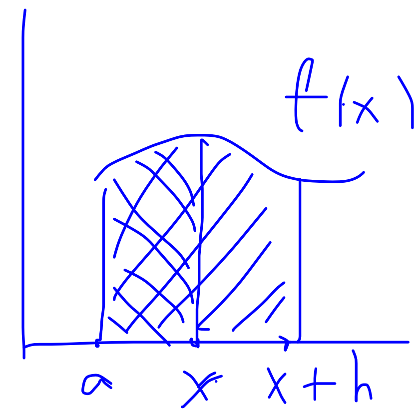
The FTC consists of two parts:

<u>Part I</u>: If f is continuous on $[a, b]$ and

$$g(x) = \int_a^x f(t)dt$$

Then g is an antiderivative of f, i.e.

$$\boxed{g' = f}$$



<u>Proof</u>:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

Proof (continued):

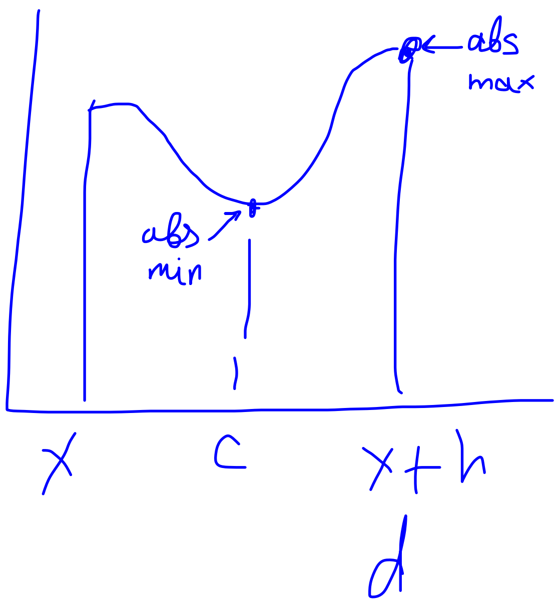
$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Let $h>0$, f is continuous on $[x, x+h]$

Then there are $c, d \in [x, x+h]$

s.t. $f(c) = m$ – abs. min

$f(d) = M$ – abs. max



$$m \le f(t) \le M$$

$$m(x+h-x) \le \int_x^{x+h} f(t)\,dt \le M(x+h-x)$$

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

Proof (continued):

$$f(c) = m \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M = f(d)$$

As $h \to 0$ $[x, x+h] \to x \Rightarrow c, d \to x$

So, as $h \to 0$, $f(c) \to f(x)$

$f(d) \to f(x)$

By the Squeeze Thm,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt \xrightarrow[h\to 0]{} f(x)$$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x) \quad \blacksquare$$

Example: Find the derivative of $g(x) = \int_3^x \overbrace{e^{t^2-t}}^{f(t)} dt$

$$g'(x) \underset{FTC, I}{=} f(x) = e^{x^2-x}$$

Example: Find the derivative of $h(x) = \int_{\sin x}^{1} \sqrt{1+t^2}\, dt$

quiz

$$= -\int_1^{\sin x} \sqrt{1+t^2}\, dt$$

If $g(x) = \int_1^x \sqrt{1+t^2}\, dt$

$$h(x) = -g(\sin x)$$

$$h'(x) = -g'(\sin x) \cdot \cos x$$

$$g'(x) \underset{FTC, I}{=} \sqrt{1+x^2} \Big| = -\sqrt{1+\sin^2 x} \cdot \cos x$$

Part II: If F is any antiderivative of f, i.e. $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note: FTC shows that differentiation and integration are inverse processes.

Proof:

$$g(x) = \int_a^x f(t)dt \underset{FTC,I}{\Longrightarrow} g'(x) = f(x)$$

g is also antiderivative of f.

So, $F(x) = g(x) + C$

$$F(b) - F(a) = g(b) + \cancel{C} - [g(a) + \cancel{C}]$$

$$= g(b) - g(a) = g(b) = \int_a^b f(t)dt$$

$$g(a) = \int_a^a f(t)dt = 0$$

$$x^n \longrightarrow \frac{1}{n+1}x^{n+1}$$

Example: Evaluate the integral $\int_0^1 x^2 dx$.

$$\int_0^1 x^2 dx = F(1) - F(0) = \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3$$

$$F(x) = \frac{1}{3}x^3 \qquad\qquad = \frac{1}{3}$$

quiz

Example: Evaluate the integral $\int_1^5 e^{x+1} dx$. $= \int_1^5 e\cdot e^x dx = e\int_1^5 e^x dx$

$$= e\,\underbrace{[e^x]}_{F(x)}\Big|_1^5 = e\cdot\underbrace{[e^5 - e^1]}_{F(5)-F(1)}$$

$$= e^6 - e^2$$

Example: Evaluate the integral $\int_1^2 \frac{1+x^2}{x^3}\,dx$. $= \int_1^2 \left[\frac{1}{x^3} + \frac{1}{x}\right] dx$

$$\frac{1}{x^3} = x^{-3} \longrightarrow \frac{x^{-3+1}}{-3+1} = -\frac{x^{-2}}{2}$$

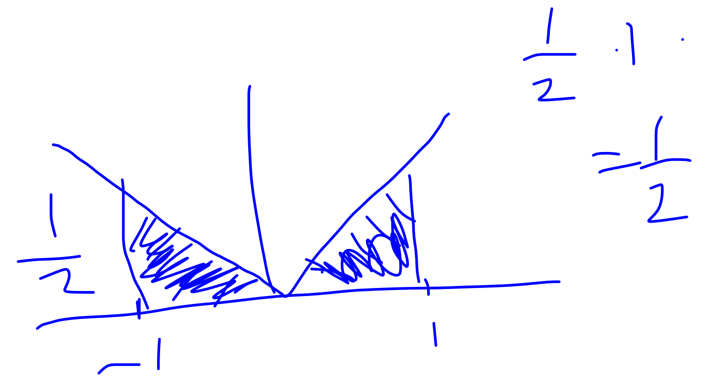
$$\frac{1}{x} \longrightarrow \ln x$$

$$= \left[-\frac{x^{-2}}{2} + \ln x\right]_1^2$$

$$= \left[-\frac{2^{-2}}{2} + \ln 2\right] - \left[-\frac{1^{-2}}{2} + \ln 1\right]$$

$$= -\frac{1}{8} + \ln 2 + \frac{1}{2} - 0 = \frac{3}{8} + \ln 2$$

Example: Evaluate the integral $\int_{-1}^{1} |x|\,dx$.



$$= \int_{-1}^{0}(-x)\,dx + \int_{0}^{1} x\,dx$$

$$|x| = \begin{cases} -x, & x \le 0 \\ x, & x \geq 0 \end{cases}$$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1}$$

$$= -\frac{0^2}{2} + \frac{(-1)^2}{2} + \frac{1^2}{2} - \frac{0^2}{2}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$