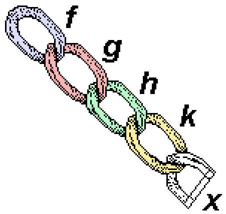
<u>Lecture 5</u> (Differentiation Rules Continued)



Chain Rule

What if we want to find the derivative of $F(x) = \sqrt{\sin x}$?

$$F(x) = (f \circ g)(x) = f(g(x))$$

$$g(x) = \sin x$$

$$f(x) = \sqrt{x}$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

Chain rule: If g is differentiable at x and f is differentiable at $g(x)$, then $F(x) = f(g(x))$ is differentiable at x and

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

(in Leibniz notation: if $y = f(x)$ and $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$)

Example: Differentiate $F(x) = \sqrt{\sin x}$

$$g(x) = \sin x \Rightarrow g'(x) = \cos x$$

$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{\sin x}} \cdot \cos x = \frac{\cos x}{2\sqrt{\sin x}}$$

<u>Example</u>: Differentiate $f(x) = e^{g(x)}$, where $g(x) = x^2 - 1$.

$$f(x) = h(g(x))$$

$$h(x) = e^x \Rightarrow h'(x) = e^x$$

$$g(x) = x^2 - 1 \Rightarrow g'(x) = 2x$$

$$f'(x) = h'(g(x)) \cdot g'(x)$$

$$= e^{g(x)} \cdot 2x$$

$$= 2x e^{x^2-1}$$

Example: Given $f(x) = (\cos^2 x - x)^{20}$, find $f'(x)$.

$$f(x) = h(g(x))$$

$$g(x) = \cos^2 x - x \Rightarrow g'(x) = (\cos^2 x)' - 1$$

$$h(x) = x^{20} \Rightarrow h'(x) = 20x^{19}$$

$$\longrightarrow \cos^2 x = F(G(x))$$

$$G(x) = \cos x \Rightarrow G'(x) = -\sin x$$

$$F(x) = x^2 \Rightarrow F'(x) = 2x$$

$$(\cos^2 x)' = F'(G(x)) \cdot G'(x)$$

$$= 2\cos x \cdot (-\sin x)$$

$$f'(x) = h'(g(x)) \cdot g'(x)$$

$$= 20(\cos^2 x - x)^{19} \cdot (-2\cos x \sin x - 1)$$

Example: Differentiate (a) $y = \sec(x^2)$ and (b) $y = \sec^2 x$

$(x^n)' = n x^{n-1}$

(a) $y = \sec(x^2) = f(g(x))$

$g(x) = x^2 \Rightarrow g'(x) = 2x$

$f(x) = \sec x \Rightarrow f'(x) = \sec x \cdot \tan x$

$y' = f'(g(x))\, g'(x) = \sec(x^2)\tan(x^2) \cdot 2x$

(b) $y = \sec^2 x = g(f(x))$

$g(x) = x^2$

$f(x) = \sec x$

$y' = g'(f(x)) \cdot f'(x) = 2\sec x \cdot \sec x \cdot \tan x$

$= 2\sec^2 x \cdot \tan x$

Power rule + Chain rule:

$F(x) = [g(x)]^n = f(g(x))$, $f(x) = x^n$

$F'(x) = f'(g(x)) \cdot g'(x)$ $\qquad f'(x) = nx^{n-1}$

$= n[g(x)]^{n-1} \cdot g'(x)$

$$\boxed{\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)}$$

Example: $y = \sec^2 x$ $= [g(x)]^2$, $g(x) = \sec x$, $n = 2$

$y' = 2[g(x)]' \cdot g'(x) = 2\sec x \cdot \sec x \cdot \tan x$

$= 2\sec^2 x \tan x$

Example: Find the derivative of $f(x) = \frac{1}{(e^x+x^2)^3}$. $= [g(x)]^n$

$g(x) = e^x + x^2$

$g'(x) = e^x + 2x$

$n = -3$

$f'(x) = (-3) \cdot (e^x + x^2)^{-3-1} \cdot (e^x + 2x)$

$= -3 \cdot (e^x + x^2)^{-4} (e^x + 2x)$

$= \frac{-3(e^x+2x)}{(e^x+x^2)^4}$

$(e^{-x})' = -e^{-x}$

$(e^{x^2})' = 2x e^{x^2}$

Example: Differentiate $y = a^x$, $a > 0$.

Recall: $a = e^{\ln a}$

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

$$(a^x)' = \left(e^{x\ln a}\right)' = e^{x\ln a} \cdot (x \ln a)'$$

$$= \ln a \cdot \left(e^{\ln a}\right)^x$$

$$= a^x \ln a$$

$$\boxed{(a^x)' = a^x \ln a}$$

Example: $y = 3^x$ $\quad a = 3$

$$y' = 3^x \ln 3$$

Chain rule for three functions and more:

$$(f \circ g \circ h)'(x) = \left[f\left(g\left(h(x)\right)\right)\right]' = f'(g\left(h(x)\right)) \cdot g'(h(x)) \cdot h'(x)$$

Proof:

$$F(x) = f(g(h(x))) = f(G(x))$$

$$G(x) = g(h(x))$$

$$F'(x) = f'(G(x)) \cdot G'(x)$$

$$G'(x) = g'(h(x)) \cdot h'(x)$$

$$F'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

<u>Example</u>: Differentiate $F(x) = e^{\sin(x^2)}$ $= f(g(h(x)))$

$h(x) = x^2 \Rightarrow h'(x) = 2x$

$g(x) = \sin x \Rightarrow g'(x) = \cos x$

$f(x) = e^x \Rightarrow f'(x) = e^x$

$F'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$

$= e^{g(h(x))} \cdot \cos(h(x)) \cdot 2x$

$= e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x$

Example: Find the derivative of $y = \cos(\tan(5^x))$ $= f(g(h(x)))$

$h(x) = 5^x \Rightarrow h'(x) = 5^x \ln 5$

$g(x) = \tan x \Rightarrow g'(x) = \sec^2 x$

$f(x) = \cos x \Rightarrow f'(x) = -\sin x$

$y' = -\sin(\tan(5^x)) \cdot \sec^2(5^x) \cdot 5^x \ln 5$

<u>Example</u>: Find $\frac{d}{dx}\left(e^{2^{x^3}}\right)$. $= f(g(h(x)))$

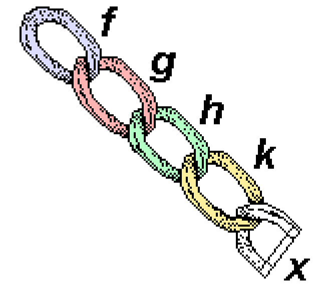
$h(x) = x^3 \Rightarrow h'(x) = 3x^2$

$g(x) = 2^x \Rightarrow g'(x) = 2^x \ln 2$

$f(x) = e^x \Rightarrow f'(x) = e^x$

$$\left(e^{2^{x^3}}\right)' = e^{2^{x^3}} \cdot 2^{x^3} \ln 2 \cdot 3x^2$$

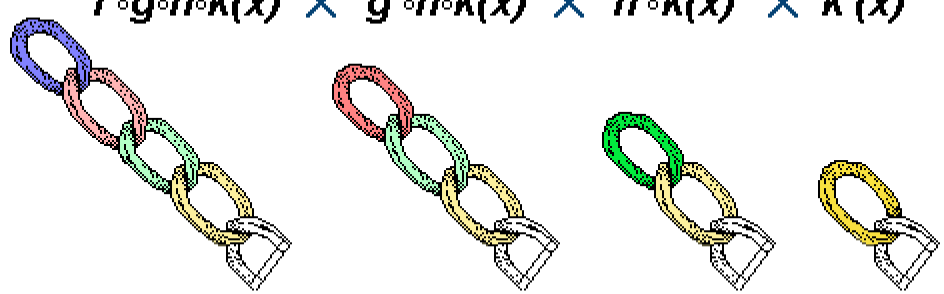
What if we have a composition of four functions?



$$(f\circ g\circ h\circ k)'(x)$$

$$= f'(g(h(k(x))))\cdot g'(h(k(x)))\cdot h'(k(x))\cdot k'(x)$$

$f'\circ g\circ h\circ k(x) \times g'\circ h\circ k(x) \times h'\circ k(x) \times k'(x)$



<u>Example</u>: $y = \left(\tan\left(\frac{1}{\sin x}\right)\right)^3 = f(g(h(k(x))))$

$$k(x) = \sin x \Rightarrow k'(x) = \cos x$$

$$h(x) = \frac{1}{x} \Rightarrow h'(x) = -\frac{1}{x^2}$$

$$g(x) = \tan x \Rightarrow g'(x) = \sec^2 x$$

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

$$y' = 3\left(\tan\frac{1}{\sin x}\right)^2 \cdot \sec^2\left(\frac{1}{\sin x}\right) \cdot \left(-\frac{1}{\sin^2 x}\right) \cdot \cos x$$

More examples:

(a) $y = \sqrt{x + \sqrt{x + \sqrt{x}}} = \sqrt{f(x)}$

$f(x) = x + \sqrt{x + \sqrt{x}} = x + g(x)$

$g(x) = \sqrt{x + \sqrt{x}} = \sqrt{h(x)}$

$h(x) = x + \sqrt{x}$

$y' = \left[\sqrt{f(x)}\right]' = \frac{1}{2}\left[f(x)\right]^{-\frac{1}{2}} \cdot f'(x)$

$f'(x) = \left[x + g(x)\right]' = 1 + g'(x)$

$g'(x) = \left[\sqrt{h(x)}\right]' = \frac{1}{2}\left[h(x)\right]^{-1/2} \cdot h'(x) = \frac{1}{2}(x + \sqrt{x})^{-1/2} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right)$

$y' = \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \cdot \left(1 + \frac{1}{2} \cdot \frac{1}{\sqrt{x + \sqrt{x}}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right)\right)$

(b) $y = (e^{-x} + x)(1 - x^2)^4$

$$y' \underset{\text{product rule}}{=} (e^{-x}+x)'(1-x^2)^4 + (e^{-x}+x)\underbrace{\left[(1-x^2)^4\right]'}_{\text{power rule + chain rule}}$$

$$= (-e^{-x}+1)(1-x^2)^4 + (e^{-x}+x)\cdot 4(1-x^2)^3\cdot(-2x)$$

$$= (1-e^{-x})(1-x^2)^4 - 2x(e^{-x}+x)(1-x^2)^3$$

Implicit Differentiation

Often we cannot express our function y in terms of x explicitly. In this case to find the derivative of y, we need to differentiate implicitly.

Let's look at the following examples.

Example: Given $x^2 + y^2 = 25$, find y'.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$$

$$2x + 2y \cdot y' = 0$$

$$y' = -\frac{x}{y}$$

$$y = f(x)$$

$$y^2 = [f(x)]^2$$

$$[y^2]' = 2f(x) \cdot f'(x)$$

$$= 2y \cdot y'$$

Find the equation of the tangent line at point (4, 3)

$$y = f(a) + f'(a)(x-a) = 3 + \left(-\frac{4}{3}\right)(x-4)$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{\substack{x=4\\y=3}} = -\frac{4}{3}$$

$$y = 3 - \frac{4x}{3} + \frac{16}{3}$$

$$\boxed{4x+3y=25}$$

Example: Find y' if $\quad 3x^4y^2 - y^4 + y + 1 = xy - x$

$$\frac{d}{dx}\left(3x^4y^2 - y^4 + y + 1\right) = \frac{d}{dx}\left(xy - x\right)$$

$$3\cdot 4x^3y^2 + \underbrace{3x^4\cdot 2yy'}_{\text{product rule}} - \underline{4y^3\cdot y'} + \underline{y'} + 0 = 1\cdot y + \underline{xy'} - 1$$

$$6x^4yy' - 4y^3y' + y' - xy' = y - 12x^3y^2 - 1$$

$$y'\left(6x^4y - 4y^3 + 1 - x\right) = y - 12x^3y^2 - 1$$

$$y' = \frac{y - 12x^3y^2 - 1}{6x^4y - 4y^3 - x + 1}$$

Example: Given $\sin(xy) + e^y = x\tan(y-1)$, find $\frac{dy}{dx}$

$$\frac{d}{dx}\left(\sin(xy) + e^y\right) = \frac{d}{dx}\left(x\tan(y-1)\right)$$

$$\cos(xy)\cdot\underbrace{(xy)'}_{1\cdot y + xy'} + e^y\cdot y' = 1\cdot\tan(y-1) + x\sec^2(y-1)\,y'$$

$$y\cos(xy) + \underline{xy'\cos(xy)} + \underline{e^y y'} = \tan(y-1) + \underline{xy'\sec^2(y-1)}$$

$$y'\left(x\cos(xy) + e^y - x\sec^2(y-1)\right) = \tan(y-1) - y\cos(xy)$$

$$y' = \frac{\tan(y-1) - y\cos(xy)}{x\cos(xy) + e^y - x\sec^2(xy)}$$

Example: Find y'' if we are given that $x^3 + y^3 = 1$

$$\frac{d^2y}{dx^2}$$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(1)$$

$$\cancel{3}x^2 + \cancel{3}y^2 \cdot y' = 0 \Longrightarrow y' = -\frac{x^2}{y^2}$$

$$\frac{d^2y}{dx^2} = y'' = \left(-\frac{x^2}{y^2}\right)' = -\frac{2x\cdot y^2 - x^2 \cdot 2yy'}{y^4}$$

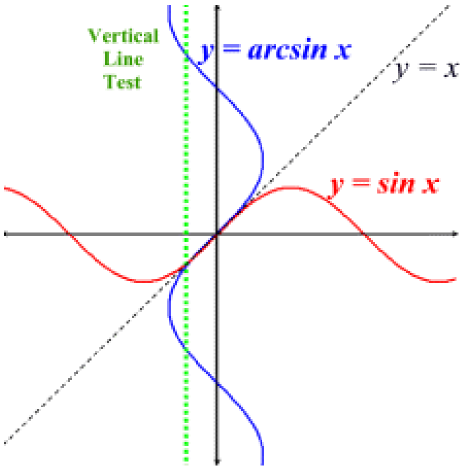
$$= -\frac{2xy^2 - 2x^2y \cdot \left(-\frac{x^2}{y^2}\right)}{y^4}$$

$$= -\frac{2xy^4 + 2x^4y}{y^6} = -\frac{2xy^3 + 2x^4}{y^5}$$

Derivatives of Inverse Trigonometric Functions

Recall the function $y = \sin^{-1} x$.

$\sin y = x$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



Now let's differentiate $\sin y = x$ implicitly with respect to x:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y \cdot y' = 1$$

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

So,

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$$

What about $y = \tan^{-1} x$?

$$\tan y = x$$

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\sec^2 y \cdot y' = 1$$

$$y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

Derivatives of Inverse Trigonometric Functions:

$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$	$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\tan^{-1} x)' = \frac{1}{1+x^2}$	$(\cot^{-1} x)' = -\frac{1}{1+x^2}$
$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$	$(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}$

Example: Find the derivative of $y = x^2 \sin^{-1}(e^{-x})$

$$y' = 2x \sin^{-1}(e^{-x}) + x^2 \cdot \frac{1}{\sqrt{1-(e^{-x})^2}} \cdot (e^{-x})'$$

$$= 2x \sin^{-1}(e^{-x}) - \frac{x^2 e^{-x}}{\sqrt{1-e^{-2x}}}$$

<u>Example</u>: $y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

$$y' = \frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2} \cdot \left(\sqrt{\frac{1-x}{1+x}}\right)'$$

$$\left[\left(\frac{1-x}{1+x}\right)^{1/2}\right]' = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \left(\frac{1-x}{1+x}\right)'$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}} \cdot \frac{(-1)(1+x)-(1-x)\cdot 1}{(1+x)^2}$$

$$= \frac{1}{\cancel{2}}\sqrt{\frac{1+x}{1-x}} \cdot \frac{-\cancel{2}}{(1+x)^2} = -\sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{(1+x)^2}$$

$$y' = -\frac{1}{1+\frac{1-x}{1+x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{1}{(1+x)^2} = -\frac{1+x}{2} \cdot \frac{(1+x)^{1/2}}{(1-x)^{1/2}(1+x)^2}$$

$$= -\frac{1}{2\sqrt{(1-x)(1+x)}} = -\frac{1}{2\sqrt{1-x^2}}$$

Derivatives of Logarithmic Functions

Fact:

$$\boxed{(\log_a x)' = \frac{1}{x \ln a}}$$

Proof:

$$y = \log_a x$$

$$a^y = x$$

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x)$$

$$a^y \ln a \cdot y' = 1$$

$$y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

$a = e$ $\quad (\ln x)' = \dfrac{1}{x \ln e} = \dfrac{1}{x}$

In particular,

$$\boxed{(\ln x)' = \frac{1}{x}}$$

<u>Example</u>: Differentiate $y = \ln(x^2 - 2x + 1)$

$$y' = \frac{1}{x^2 - 2x + 1}(x^2 - 2x + 1)'$$

$$= \frac{1}{(x-1)^2}(2x - 2)$$

$$= \frac{2(x-1)}{(x-1)^2} = \frac{2}{x-1}$$

Example: Find the derivative of $f(x) = (\log_2 x^2)^3$

$$f'(x) = 3(\log_2 x^2)^2 \cdot (\log_2 x^2)'$$

$$(\log_2 x^2)' = \frac{1}{x^2 \ln 2} \cdot 2x$$

$$f'(x) = 3(\log_2 x^2)^2 \cdot \frac{2x}{x^2 \ln 2}$$

$$= \frac{6(\log_2 x^2)^2}{x \ln 2}$$

Example: Given $f(x) = \ln|x|$, find $f'(x)$.

$$f(x) = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x}, & x < 0 \end{cases}$$

$$f'(x) = \frac{1}{x}, \quad x \neq 0$$

Logarithmic Differentiation

Sometimes taking logarithms of functions can simplify the calculation of derivatives.

Example: Differentiate $y = \frac{\sqrt[3]{x-1}}{x^2(x+2)^7}$

$$\ln y = \ln\left(\frac{\sqrt[3]{x-1}}{x^2(x+2)^7}\right)$$

$$(\ln y)' = \left(\ln (x-1)^{1/3} - \ln x^2 - \ln (x+2)^7\right)'$$

$$= \left(\frac{1}{3}\ln(x-1) - 2\ln x - 7\ln(x+2)\right)'$$

$$\frac{1}{y}\cdot y' = \frac{1}{3}\,\frac{1}{x-1}\cdot 1 - 2\,\frac{1}{x} - 7\,\frac{1}{x+2}\cdot 1$$

$$y' = \left[\frac{1}{3(x-1)} - \frac{2}{x} - \frac{7}{x+2}\right]y = \left[\frac{1}{3(x-1)} - \frac{2}{x} - \frac{7}{x+2}\right]\cdot\frac{\sqrt[3]{x-1}}{x^2(x+2)^7}$$

Logarithmic Differentiation:

- Take natural logarithms of both sides of an equation $y = f(x)$. Simplify if possible.
- Differentiate implicitly with respect to x.
- Solve for y'.

Example: Differentiate $y = x^{\ln x}$

$$\ln y = \ln x^{\ln x} = (\ln x)(\ln x) = (\ln x)^2$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[(\ln x)^2\right]$$

$$\frac{1}{y} \cdot y' = 2 \ln x \cdot \frac{1}{x}$$

$$y' = \frac{2 \ln x}{x} y = \frac{2 \ln x}{x} x^{\ln x}$$

Back to number *e*

$$f(x) = \ln x \implies f'(x) = \frac{1}{x} \implies f'(1) = 1$$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$1 = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{\ln(1+h)}{h}$$

(took x instead of h)

$$e = e^1 = e^{\lim_{x\to 0} \frac{\ln(1+x)}{x}} = e^{\lim_{x\to 0} \frac{1}{x}\ln(1+x)}$$

$$= e^{\lim_{x\to 0} \ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0} (1+x)^{1/x}$$

$$\boxed{e = \lim_{x \to 0} (1+x)^{1/x}}$$

x	$(1+x)^{1/x}$
0.1	2.593742
0.01	2.704813
0.001	2.716923
0.0001	2.718145
0.00001	2.718268
0.000001	2.718280
0.0000001	2.718281

$\Rightarrow e \approx 2.718$

Take $n = \frac{1}{x} \Rightarrow n \to \infty$
$x \to 0^+$

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

Example: Show that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for any $x>0$.

$$\left(\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k\right)^x = e^x$$

$$\lim\left(1+\frac{1}{k}\right)^{kx} = e^x$$

Take $n = kx \Rightarrow k = \frac{n}{x} \xrightarrow[n\to\infty]{} \infty$

$$\frac{x}{n} = \frac{1}{k}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{kx} = e^x \quad \blacksquare$$