Lecture 4 (Differentiation Rules)

Recall:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

What is the derivative of a constant? I.e. let $f(x) = c$ for all x.

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

Thus,

$$\boxed{\frac{d}{dx}(c) = 0}$$

What about $f(x) = x$?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h} = \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1 = 1$$

$$\boxed{\frac{d}{dx}(x) = 1}$$

We can show that

$$\boxed{\frac{d}{dx}(x^2) = 2x}$$ $\longleftarrow$ exe

What if $f(x) = x^3$?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim \frac{\cancel{x^3} + 3x^2 \cancel{h} + 3xh^2 + h^3 - \cancel{x^3}}{\cancel{h}}$$

$$= \lim_{h \to 0} [3x^2 + 3xh + h^2] = 3x^2$$

$$\boxed{\frac{d}{dx}(x^3) = 3x^2}$$

$$(x^n)' = nx^{n-1}$$

In general, if $n \in \mathbb{Z}^+$, then

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

<u>Proof</u>: Binomial formula:

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n+1)}{2}x^{n-2}y^2 + \dots + nxy^{n-1} + y^n$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{x^n} + nx^{n-1}\cancel{h} + \frac{n(n+1)}{2}x^{n-2}h^{\cancel{2}} + \dots + nxh^{\cancel{n-1}} + h^{\cancel{n}} - \cancel{x^n}}{\cancel{h}}$$

$$= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n+1)}{2}x^{n-2}h + \dots + nxh^{n-2} + h^{n-1}\right]$$

$$= nx^{n-1}$$

$n=5$

Example: Given $f(x)=x^5$, find $f'''(2)$.

$f'(x) = 5x^{5-1} = 5x^4$

$f''(x) = (5x^4)' = 5(x^4)' = 5\cdot 4x^3$
$= 20x^3$

$[c\,g(x)]' = c\,g'(x)$

$f'''(x) = (20x^3)' = 20(x^3)'$
$= 20\cdot 3x^2$
$= 60x^2$

$f'''(2) = 60\cdot 2^2 = 60\cdot 4 = 240$

General Power Rule: If $n \in \mathbb{R}$ (any real number), then

$$\left(x^{-\frac{1}{3}}\right)' = -\frac{1}{3} \cdot x^{-\frac{1}{3}-1}$$
$$= -\frac{1}{3} x^{-\frac{4}{3}}$$
$$= \frac{-1}{3\sqrt[3]{x^4}}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$\left(\sqrt[3]{x} - \sqrt[3]{x+h}\right)\left(\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x(x+h)} + \left(\sqrt[3]{x+h}\right)^2\right)$$

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

Example: Find derivative of $f(x) = \frac{1}{\sqrt[3]{x}}$. $= x^{-1/3}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt[3]{x+h}} - \frac{1}{\sqrt[3]{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{x+h}}{h\sqrt[3]{x}\sqrt[3]{x+h}} \cdot \frac{\left(\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}\sqrt[3]{x+h} + \left(\sqrt[3]{x+h}\right)^2\right)}{\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}\sqrt[3]{x+h} + \left(\sqrt[3]{x+h}\right)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - (\cancel{x} + \cancel{h})}{\cancel{h}\sqrt[3]{x}\sqrt[3]{x+h}\left(\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}\sqrt[3]{x+h} + \left(\sqrt[3]{x+h}\right)^2\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt[3]{x}\sqrt[3]{x+h}\left(\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{x}\sqrt[3]{x+h} + \left(\sqrt[3]{x+h}\right)^2\right)} = \frac{-1}{3\sqrt[3]{x^4}}$$

Basic Rules of Differentiation:

- *Constant multiple rule*:

$$\boxed{[cf(x)]' = cf'(x)}$$

Proof:

$$[cf(x)]' = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c\, f'(x)$$

- *Sum/difference rule*:

$$\boxed{[f(x) \pm g(x)]' = f'(x) \pm g'(x)}$$

<u>Proof</u>:

$$[f(x) + g(x)]' = \lim_{h \to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x) \qquad \blacksquare$$

Example: Given $f(x) = 2x^2 - \frac{1}{x^3} - 1$. Find $f'(x)$.

$$f'(x) = (2x^2)' - \left(\frac{1}{x^3}\right)' - (1)'$$

$$= 2 \cdot 2x - (x^{-3})' - 0$$

$$= 4x + 3x^{-3-1}$$

$$= 4x + 3\frac{1}{x^4}$$

<u>Example</u>: Given $f(x) = x^3 - 2x^2 + 3x - 1$, find $f^{(4)}(x)$.

$$f'(x) = 3x^2 - 2\cdot 2x + 3\cdot 1 - 0$$

$$= 3x^2 - 4x + 3$$

$$f''(x) = 3\cdot 2x - 4\cdot 1 + 0$$

$$= 6x - 4$$

$$f'''(x) = 6\cdot 1 - 0$$

$$= 6$$

$$f^{(4)}(x) = 0$$

Now consider exponential function $f(x) = a^x$. What is $f'(x)$?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{f'(0)}$$

$$= a^x f'(0)$$

$$\boxed{f'(x) = f'(0)a^x}$$

Consider two cases: $a = 2$ and $a = 3$.

$e = 2.7$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987
	<1	>1

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

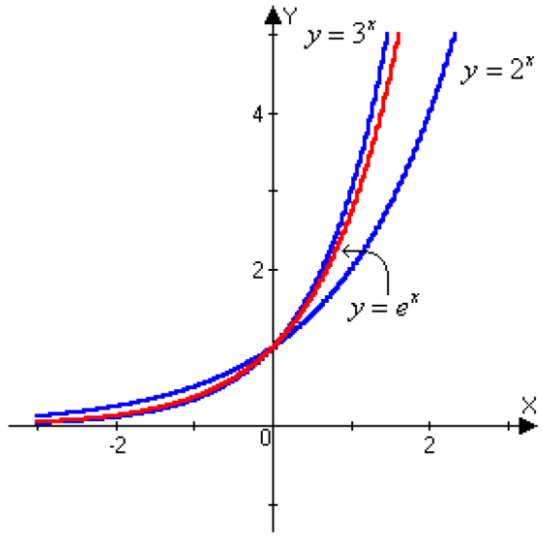
If $f(x) = e^x$

$f'(x) = e^x f'(0) = e^x$

Definition of Number *e*:

e is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$



Thus,

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Example: Find the tangent line to $y = 2x^2 + e^x$ at $(0, 1)$.

$$y = f(a) + f'(a)(x-a)$$

$$= 1 + f'(0)(x-0)$$

$$= 1 + f'(0)x$$

$$f'(x) = (2x^2+e^x)' = 2 \cdot 2x + e^x$$

$$= 4x + e^x$$

$$f'(0) = 4 \cdot 0 + e^0 = 1$$

So, $y = 1 + f'(0)x = 1 + 1 \cdot x = 1 + x$

$$\boxed{y = x+1}$$

Example: $f(x) = 3,\ g(x) = x^2$. Does $[f(x)g(x)]' = f'(x)g'(x)$? No

$f'(x) = (3)' = 0$

$g'(x) = (x^2)' = 2x$

$f'(x)\,g'(x) = 0 \cdot 2x = 0$

$\neq$

$[f(x)g(x)]' = (3x^2)' = 3 \cdot 2x = 6x$

- *Product rule*:

$$\boxed{[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)}$$

<u>Proof</u>:

$$[f(x)g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \qquad \pm f(x+h)g(x)$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \underbrace{f(x+h)}_{f(x)} \underbrace{\frac{[g(x+h) - g(x)]}{h}}_{g'(x)} + \lim_{h\to 0} g(x) \underbrace{\frac{[f(x+h) - f(x)]}{h}}_{f'(x)}$$

$$= f(x)g'(x) + f'(x)g(x) \qquad \blacksquare$$

<u>Example</u>: Given $f(x) = xe^x$, find $f^{(n)}(x)$

$$f'(x) = x \cdot e^x + 1 \cdot e^x = \underbrace{e^x}\,\underbrace{(x+1)}$$

$$f''(x) = e^x \cdot 1 + e^x(x+1) = \underbrace{e^x}\,\underbrace{(x+2)}$$

$$f'''(x) = e^x \cdot 1 + e^x(x+2) = \underbrace{e^x}\,\underbrace{(x+3)}$$

$$f^{(4)}(x) = e^x(x+4)$$

$\vdots$

$$f^{(n)}(x) = e^x(x+n)$$

- *Quotient rule*: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g\prime(x)}{[g(x)]^2}$

$g(x) \neq 0$

Proof:

$$\left[\frac{f(x)}{g(x)}\right]' = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

$$= \lim_{h\to 0} \frac{\boxed{f(x+h)}g(x) - f(x)g(x+h) \pm f(x)g(x)}{h\,g(x)\,g(x+h)}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x)}{h\,g(x)\,g(x+h)}$$

$$= \lim_{h\to 0} \underbrace{\frac{1}{g(x)g(x+h)}}_{\to g(x)} \left[g(x) \underbrace{\frac{f(x+h)-f(x)}{h}}_{f'(x)} - f(x) \underbrace{\frac{g(x+h)-g(x)}{h}}_{g'(x)} \right]$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

∎

Example: Let $y = \frac{x^2-1}{\sqrt{x}}$, find $\frac{dy}{dx}$.

$$y = \frac{x^2}{\sqrt{x}} - \frac{1}{\sqrt{x}} = x^{\frac{3}{2}} - x^{-\frac{1}{2}}$$

$$y' = \frac{3}{2} x^{\frac{3}{2}-1} + \frac{1}{2} x^{-\frac{1}{2}-1} = \underline{\frac{3}{2} x^{\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}}}$$

$$f(x) = x^2 - 1, \quad g(x) = \sqrt{x}$$

$$y' = \frac{f'g - fg'}{g^2} = \frac{2x \cdot \sqrt{x} - (x^2-1)\frac{1}{2}\frac{1}{\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{4x^2 - (x^2-1)}{2x\sqrt{x}} = \underline{\frac{3x^2+1}{2x\sqrt{x}}}$$

Example: Find the derivative of $h(x) = \frac{e^x - x^2}{e^x}$. $= 1 - \frac{x^2}{e^x}$

$$h'(x) = 0 - \left(\frac{x^2}{e^x}\right)'$$

$$= -\frac{2x\,e^x - x^2 \cdot e^x}{e^{2x}} = \frac{\cancel{e^x}\,x(x-2)}{e^{2\cancel{x}}}$$

$$= \frac{x(x-2)}{e^x}$$

Example: If $h(2) = 4$ and $h'(2) = -3$, find $\frac{d}{dx}\left(\frac{h(x)}{x}\right)\Big|_{x=2}$.

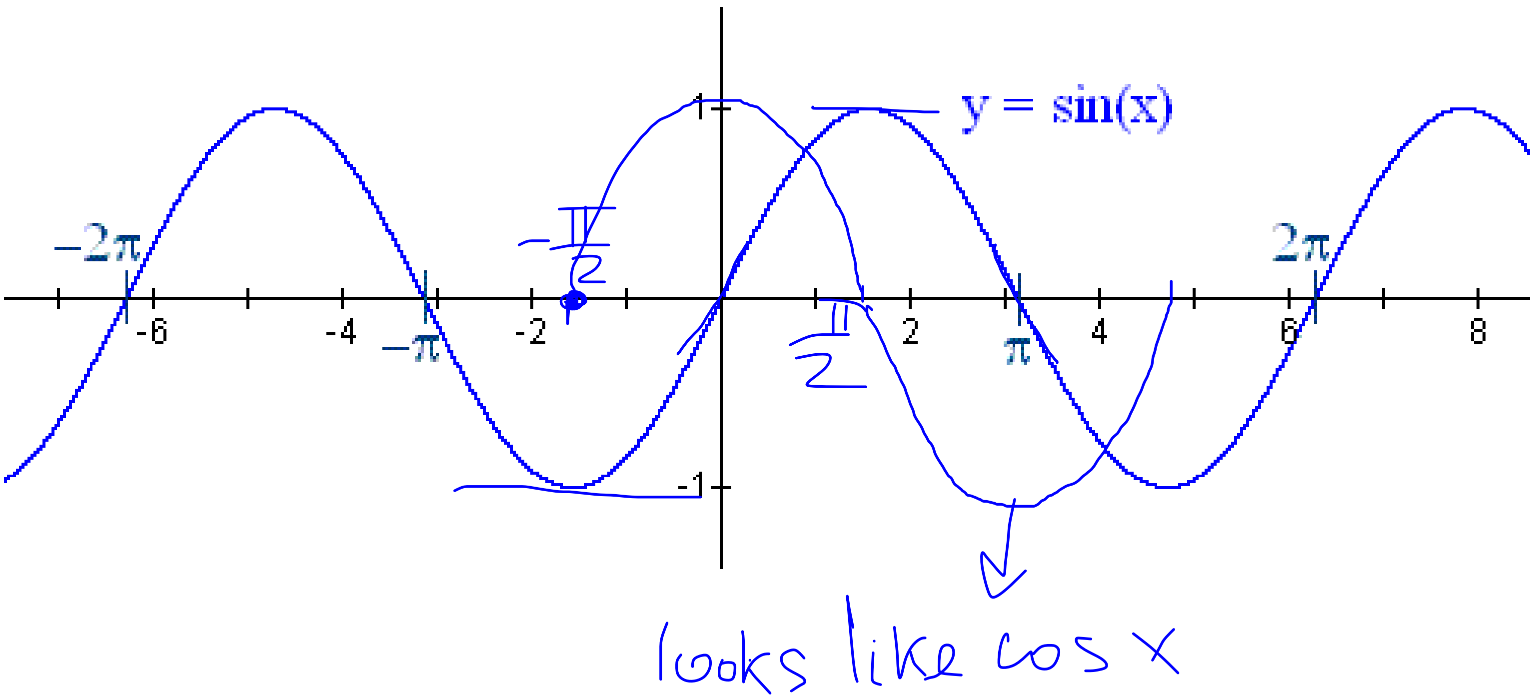
$$\left[\frac{h(x)}{x}\right]' = \frac{h'(x)x - h(x)\cdot 1}{x^2}\Bigg|_{x=2}$$

$$= \frac{h'(2)\cdot 2 - h(2)}{2^2}$$

$$= \frac{-3\cdot 2 - 4}{4} = -\frac{10}{4} = -2.5$$

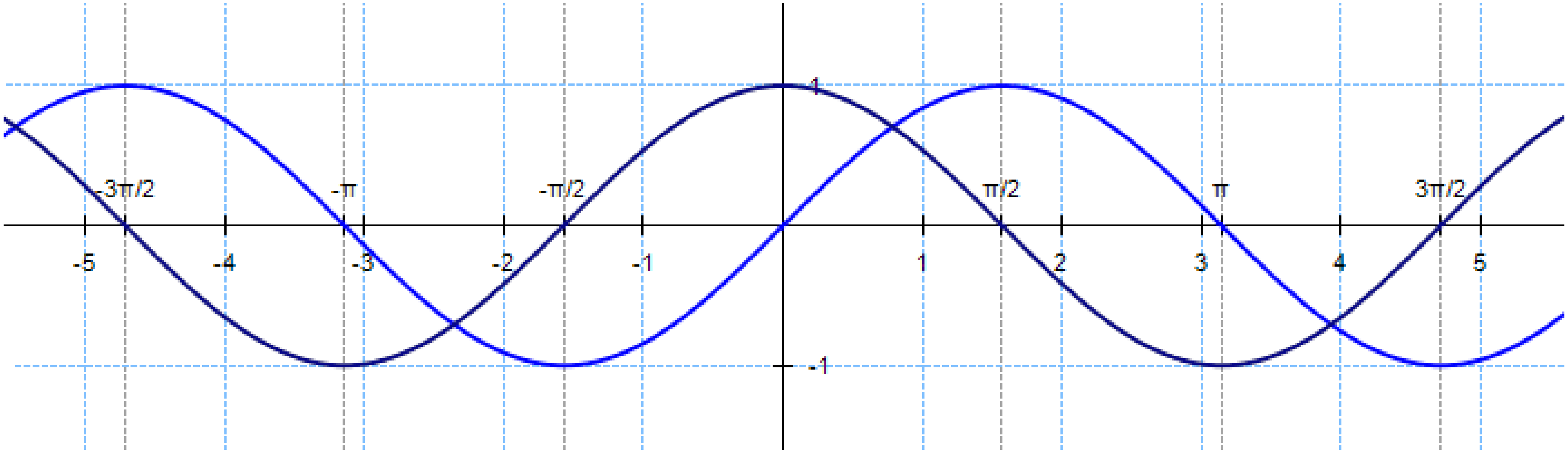
Differentiating Trigonometric Functions

Let's consider $f(x) = \sin x$



What does the graph of its derivative look like?

The graph of $f'(x) = \sin x$ looks like $\cos x$.



Let's try to prove that!

Before we do that, we shall need the following limits:

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

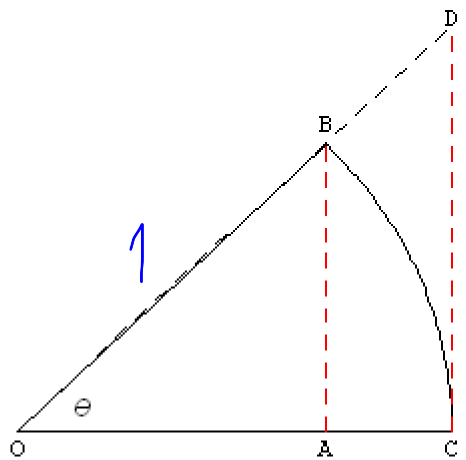
and

$$\boxed{\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0}$$

<u>Fact 1</u>: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

<u>Proof</u>: Consider an arc of circle with $r = 1$

$OB = OC = 1$



Area of circle $= \pi r^2 = \pi$

Area of sector $(BOC) = \frac{1}{2} r^2 \theta = \frac{\theta}{2}$

$OA = OB\cos\theta = \cos\theta$

$AB = OB\sin\theta = \sin\theta$

$CD = OC\tan\theta = \tan\theta$

Area ΔAOB < area of sector

$\frac{1}{2} OA \cdot AB < \frac{\theta}{2}$

$\frac{1}{2}\cos\theta\sin\theta < \frac{\theta}{2}$

$\frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}$

area of sector < area ΔCOD

$\frac{\theta}{2} < \frac{1}{2} CD \cdot OC$

$\frac{\theta}{2} < \frac{1}{2}\tan\theta = \frac{1}{2}\frac{\sin\theta}{\cos\theta}$

$\cos\theta < \frac{\sin\theta}{\theta}$

$1 \xleftarrow{\theta\to 0} \cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta} \xrightarrow{\theta\to 0} 1$ | By the Squeeze Thm, $\frac{\sin\theta}{\theta} \xrightarrow{\theta\to 0} 1$

Fact 2: $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

Proof:

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1}$$

$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)}$$

$$= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\cos\theta + 1}$$

$$= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin\theta}{\cos\theta + 1} = -1 \cdot \frac{0}{2} = 0$$

$\blacksquare$

<u>Claim</u>: $(\sin x)' = \cos x$

<u>Proof</u>:

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\sin x \underbrace{\frac{(\cos h - 1)}{h}}_{\to 0} + \cos x \underbrace{\frac{\sin h}{h}}_{\to 1} \right]$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

$\blacksquare$

Similarly, we can show that

$$(\cos x)' = -\sin x$$

What about $f(x) = \tan x$? $\tan x = \dfrac{\sin x}{\cos x}$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Derivative of Trigonometric Function

$(\sin x)' = \cos x$ $\qquad$ $(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$ $\qquad$ $(\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x \tan x$ $\qquad$ $(\csc x)' = -\csc x \cot x$

$$\sec^2 x - \tan^2 x = \frac{1}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}$$
$$= 1$$

Example: Differentiate $f(x) = \frac{\tan x}{1+\sec x}$.

$$f'(x) = \frac{\sec^2 x\,(1+\sec x) - \tan x \cdot \sec x \cdot \tan x}{(1+\sec x)^2}$$

$$= \frac{\sec^2 x + \sec^3 x - \sec x \cdot \tan^2 x}{(1+\sec x)^2}$$

$$= \frac{\sec^2 x + \sec x\,(\sec^2 x - \tan^2 x)}{(1+\sec x)^2} = \frac{\sec^2 x + \sec x}{(1+\sec x)^2} = \frac{\sec x}{1+\sec x}$$

<u>Example</u>: Find 32^{th} derivative of $\sin x$. $= f(x)$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f^{(5)}(x) = \cos x$

$f^{(6)}(x) = -\sin x$

$f^{(7)}(x) = -\cos x$

$f^{(8)}(x) = \sin x$

So, $f^{(32)}(x) = \sin x$

Example: Find $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 2x}$

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

$$\Downarrow$$

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$$

$\theta = 5x$ $\qquad \dfrac{\sin 5x}{5x} \cdot 5x$

$\theta = 2x$ $\qquad \dfrac{2x}{\sin 2x} \cdot \dfrac{1}{2x}$

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 2x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \frac{2x}{\sin 2x} \cdot \frac{5\cancel{x}}{2\cancel{x}}$$

$$= \frac{5}{2} \lim_{\substack{5x\to 0\\ \theta = 5x}} \frac{\sin 5x}{5x} \lim_{\substack{2x\to 0\\ \theta = 2x}} \frac{2x}{\sin 2x} = \frac{5}{2} \cdot 1 \cdot 1$$

$$= 2.5$$