Lecture 3 (Limits and Derivatives)

Continuity

In the previous lecture we saw that very often the limit of a function $f(x)$ as $x \to a$ is just $f(a)$. When this is the case we say that $f(x)$ is *continuous* at a.

Definition: A function f is **continuous at *a*** if

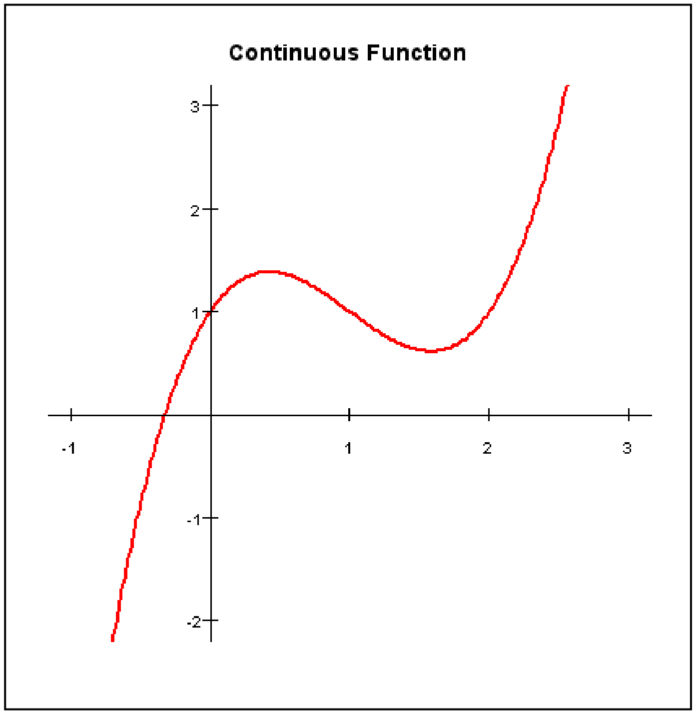
$$\lim_{x \to a} f(x) = f(a)$$

To verify continuity, we need to check three things:

1. $f(a)$ is defined

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

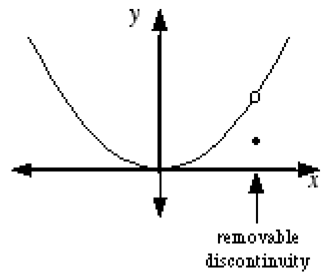


We say f is **discontinuous at *a*** if f is not continuous at a.

<u>Types of discontinuity</u>:

- removable discontinuity

$$\lim_{x \to a} f(x) \neq f(a)$$

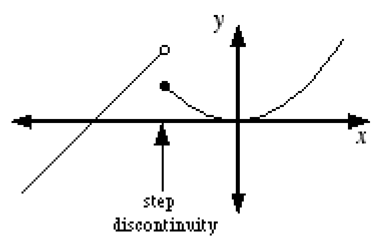


- step or jump discontinuity

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

$\lim_{x \to a} f(x)$ dne

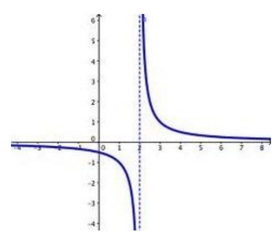
$f(a)$ is defined



- infinite discontinuity

$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = \infty$$



Example[1]: For each function, find all points of discontinuity and classify them.

(a) $f(x) = \begin{cases} 2x+4, & x<2 \\ 7, & x=2 \\ x^3, & x>2 \end{cases}$

Check $x=2$.

$f(2) = 7$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+4) = 8$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3 = 8$

$\lim_{x\to 2} f(x) = 8$

$\neq f(2) = 7$

So $x=2$ is removable discontinuity

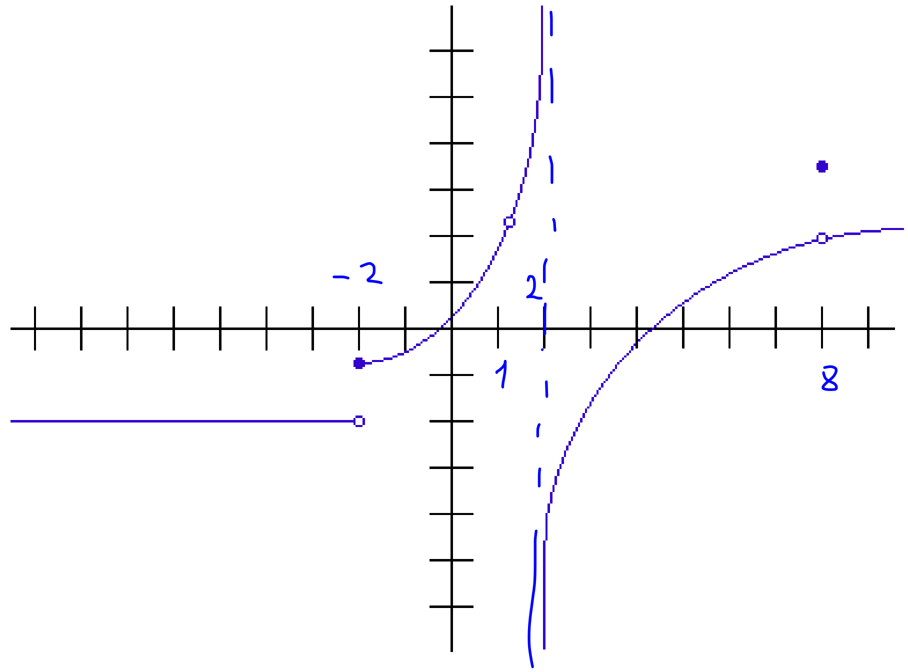
[1] http://oregonstate.edu/instruct/mth251/cq/Stage4/Practice/classify.html

(b)

$x = -2$ is a jump

$x = 1$ and $x = 8$
are removable

$x = 2$ is infinite discontinuity

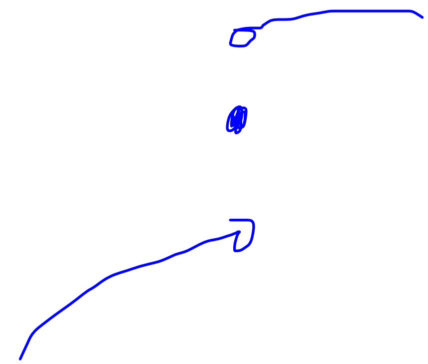


Definition: We say f is **continuous from the right at a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

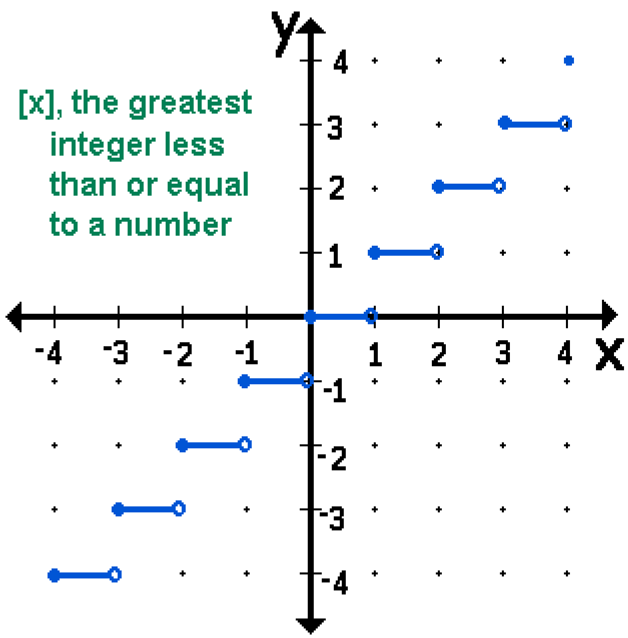


Example: $f(x) = [x]$

$f(3.1) = 3$

$f(4.9) = 4$

$f(5) = 5$



$f(x) = 2, \ x \in [2, 3)$

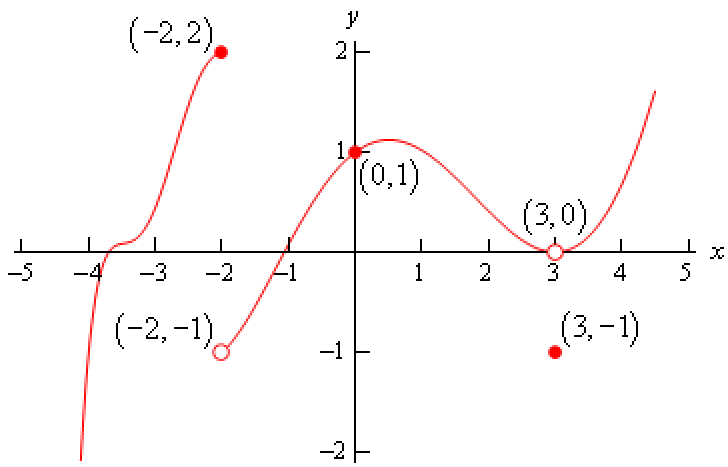
$f(x)$ is right continuous at $x = k, \ k \in \mathbb{Z}$
$= \ldots, -1, 0, 1, 2, \ldots$

but discontinuous at $x = k, \ k \in \mathbb{Z}$
(jump discontinuity)

Definition: A function f is said to be **continuous on an interval (a, b)**, if it is continuous at every number $c \in (a, b)$.

Example: Is this function continuous on interval

(a) (-4, -2)? Yes
(b) (-3, 0)? No $x=-2 \to$ jump
(c) (0, 4)? No $x=3 \to$ removable



Example: $f(x) = x^2$ is continuous on $(-\infty, +\infty)$ or everywhere.

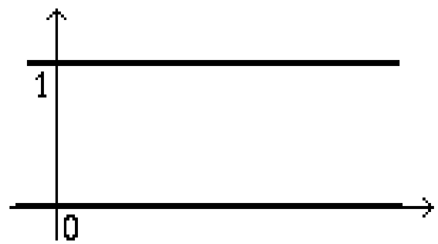
$f(x) = \frac{1}{x-1} \to$ discont. at $x=1$ (infinite)

discontinuous at $x = -1$ (removable) $\leftarrow f(x) = \frac{x^2-1}{x+1} = \frac{(x-1)(x+1)}{x+1} = x-1$

$$D(x) = \begin{cases} c, & x \in \mathbb{Q} \\ d, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Example[2] (Dirichlet function): $d(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - \mathbb{Q} \end{cases}$

Note: both rational numbers (where $d = 1$) and irrational numbers (where $d = 0$) are dense in the real line. This means that if we take any open interval, however small, there will be some rational and irrational number in this interval.



Also, there is not a single point where we would have any one-sided limit.

Indeed, pick some number a. In any neighborhood of a we find both rational and irrational numbers, so on this neighborhood d oscillates between 0 and 1. So there is no limit at a, not even one-sided.

We say, $d(x)$ is a function which is **discontinuous at every point**.

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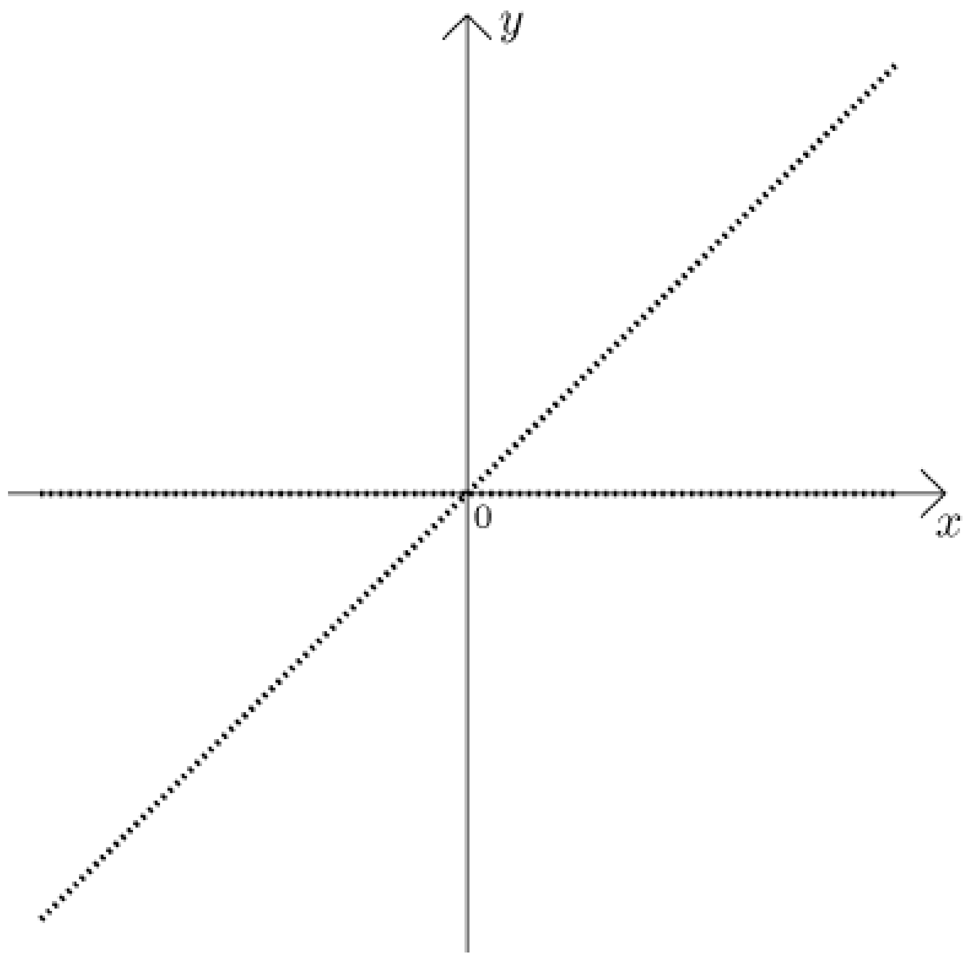
2 http://math.feld.cvut.cz/mt/txtb/4/txe3ba4s.htm

Define $f(x) = d(x) \cdot x = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - \mathbb{Q} \end{cases}$



Continuity Laws

If f and g are continuous at a and c is a constant, then

- $f \pm g$ is continuous at a
- cf is continuous at a
- fg is continuous at a
- $\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$

Theorem: The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions/inverse trigonometric functions
- Exponential functions/logarithmic functions

Example: Find the intervals of continuity for the following function

$$f(x) = \frac{\sin x + \ln(1-x)}{1-x^2}$$

$\sin x$ is cont. on $(-\infty, \infty)$

$\ln(1-x)$ is cont. on $(-\infty, 1)$

domain for $\ln(1-x)$

Also $x \neq \pm 1$

So, $f(x)$ is cont. on $(-\infty, -1) \cup (-1, 1)$

Theorem: If g is continuous at a and f is continuous at $g(a)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example: Define where the following functions are continuous.

(a) $h(x) = \cos(x^3)$

$g(x) = x^3$ is cont. on $(-\infty, \infty)$

$f(x) = \cos x$ is cont. on $(-\infty, \infty)$

$\Rightarrow h(x) = f(g(x))$ is cont. on $(-\infty, \infty)$

(b) $h(x) = \ln(1 - \sin x)$

$1 - \sin x > 0$

$\sin x < 1$

So $\sin x \neq 1$

Domain $= \{x : x \neq \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\}$

$h(x)$ is continuous on intervals $\left(-\frac{3\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right)$

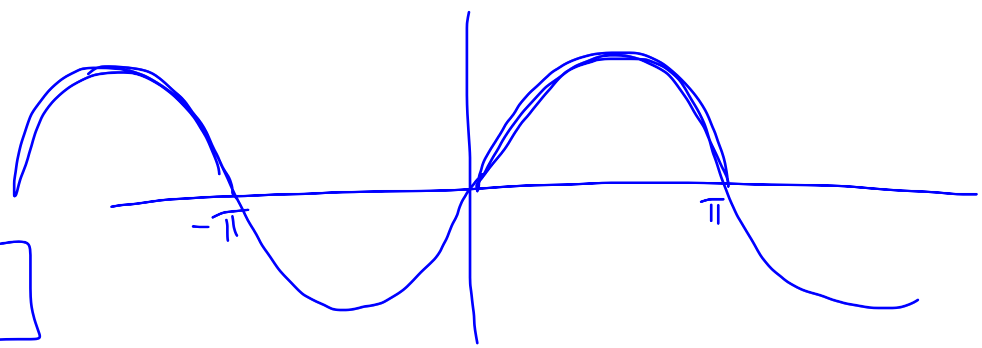
(c) $h(x) = \sqrt{\sin x}$

Domain $= \{x : \sin x \geq 0\}$

$= [0 + 2k\pi, \pi + 2k\pi]$

$= [2k\pi, (1+2k)\pi]$

$k \in \mathbb{Z}$

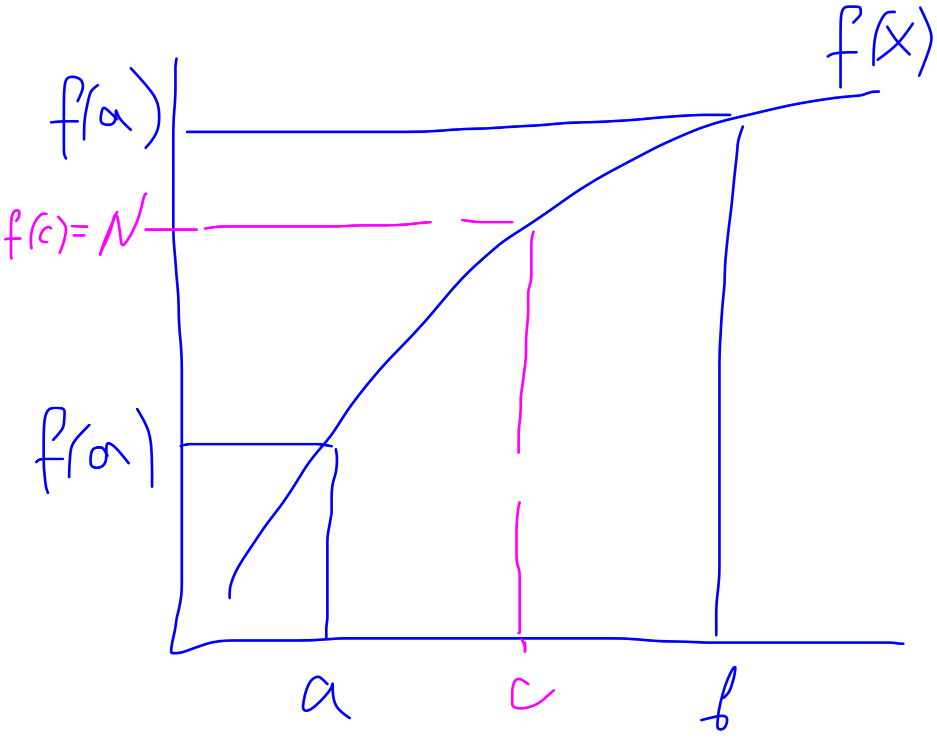


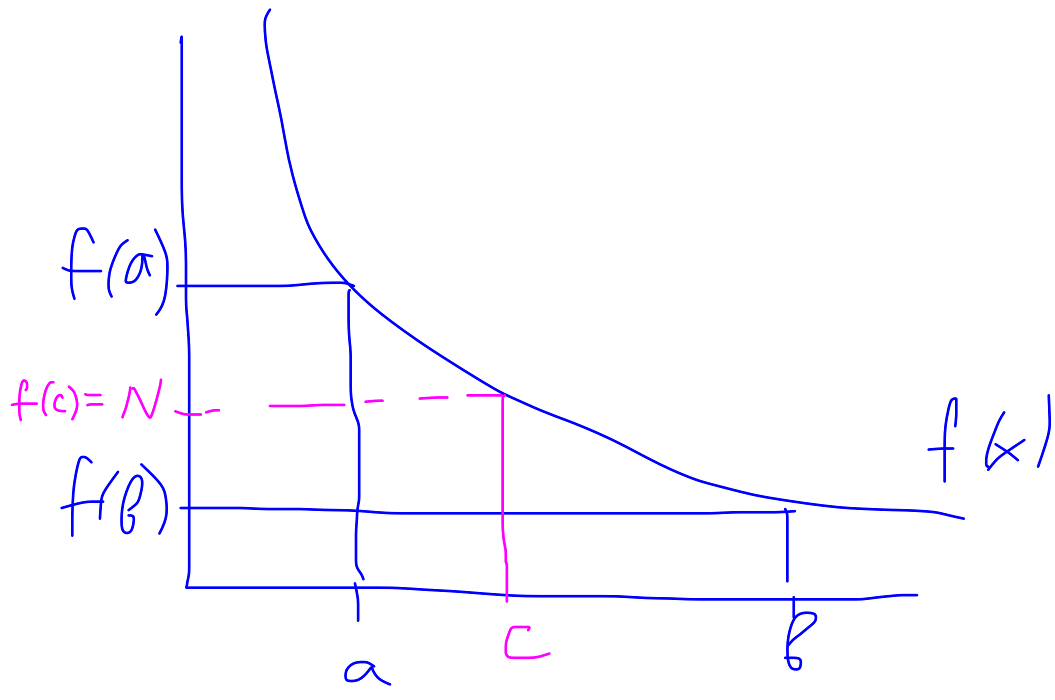
Intermediate Value Theorem (IVT): Let f be continuous on $[a, b]$ and N be such that

$$f(a) < N < f(b), \quad f(a) \neq f(b)$$

$$(\text{ or } f(a) > N > f(b) \ )$$

Then there exists $c \in (a, b)$ such that $f(c) = N$.





a, b

Example[3]: Show that $2x^3 - 5x^2 - 10x + 5 = 0$ has a root in $[-1, 2]$

$f(x) = 2x^3 - 5x^2 - 10x + 5$

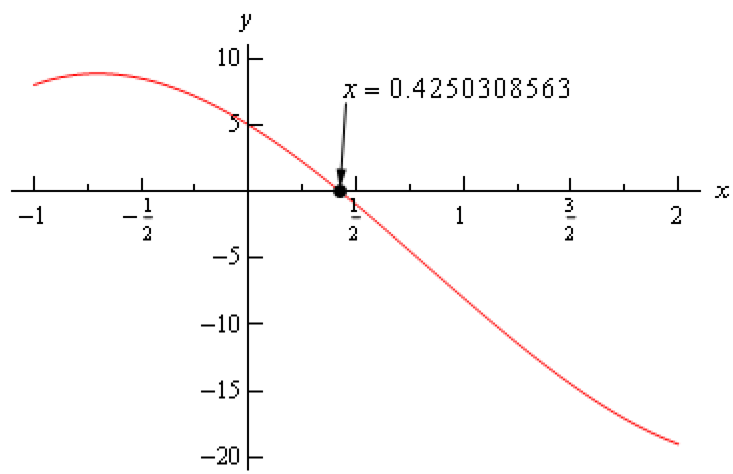
$N = 0$

$f(a) = f(-1) = 2(-1)^3 - 5(-1)^2 - 10(-1) + 5 = 8$

$f(b) = f(2) = 2 \cdot 2^3 - 5 \cdot 2^2 - 10 \cdot 2 + 5 = -19$

$-19 \le 0 \le 8 \Rightarrow$ there is $c \in [-1, 2]$ s.t. $f(c) = 0$

Here is a graph showing the root that we just proved existed.



Note: we used a computer program to actually find the root, the IVT did not tell us what this value was.

[3] http://tutorial.math.lamar.edu/Classes/CalcI/Continuity.aspx#Limit_Cont_Ex5a

Example[4]: If possible, determine if $f(x) = 20\sin(x+3)\cos\frac{x^2}{2}$ takes the following values in the interval $[0,5]$.

(a) Does $f(x) = 10$?

$[a, b] = [0, 5]$

$f(0) = 20 \sin(3) \cos\frac{0^2}{2} = 20 \sin(3)$

$= 2.8224$

$f(5) = 20 \sin 8 \cos\frac{25}{2} = 19.7436$

$2.82 \leq 10 \leq 19.74$

By IVT,

there is $c \in [0,5]$ s.t.

$f(c) = 10$

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4 http://tutorial.math.lamar.edu/Classes/CalcI/Continuity.aspx#Limit_Cont_Ex5a

(b) Does $f(x) = -10$?

-10 is not between

$f(0)$ and $f(5)$

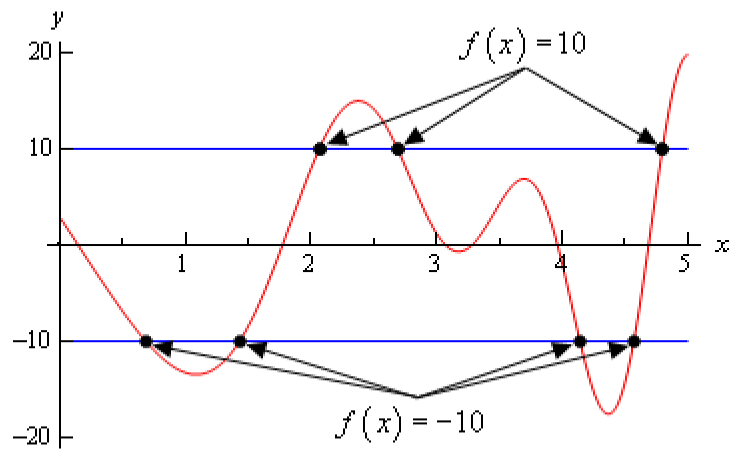
Does it mean that $f(x) \neq -10$ in $[0, 5]$?

No!

We cannot apply IVT

Note: The IVT will only tell us that c's will exist. The theorem will NOT tell us that c's don't exist.

Here is the graph of $f(x)$:



From the graph we see that not only does $f(x) = -10$ in [0,5] it does so 4 times! We also verified that $f(x) = 10$ in [0,5] and in fact it does so 3 times.

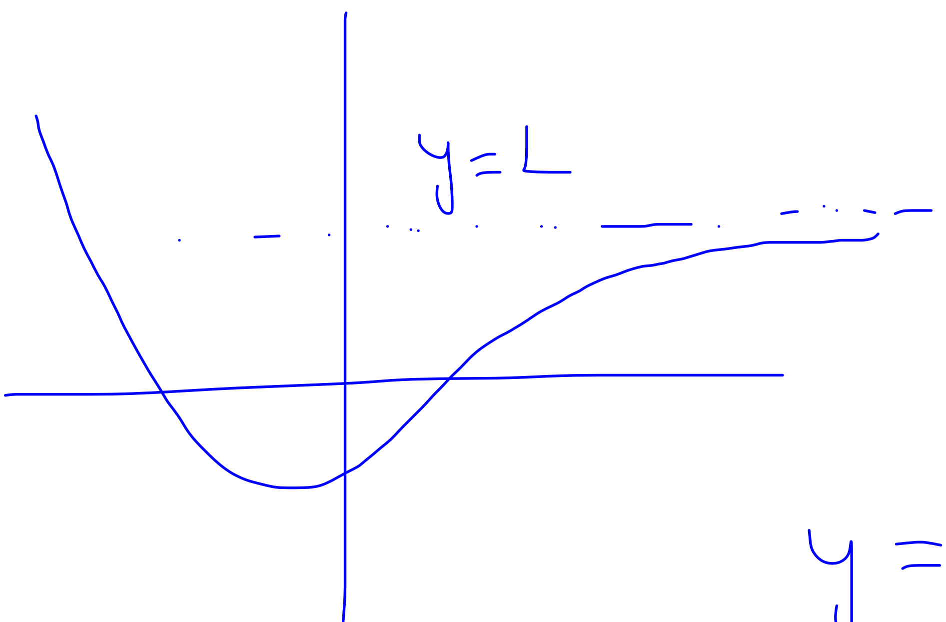
<u>Note</u>: We can use the IVT to verify that a function will take on a value, but it never tells us how many times the function will take on that value.

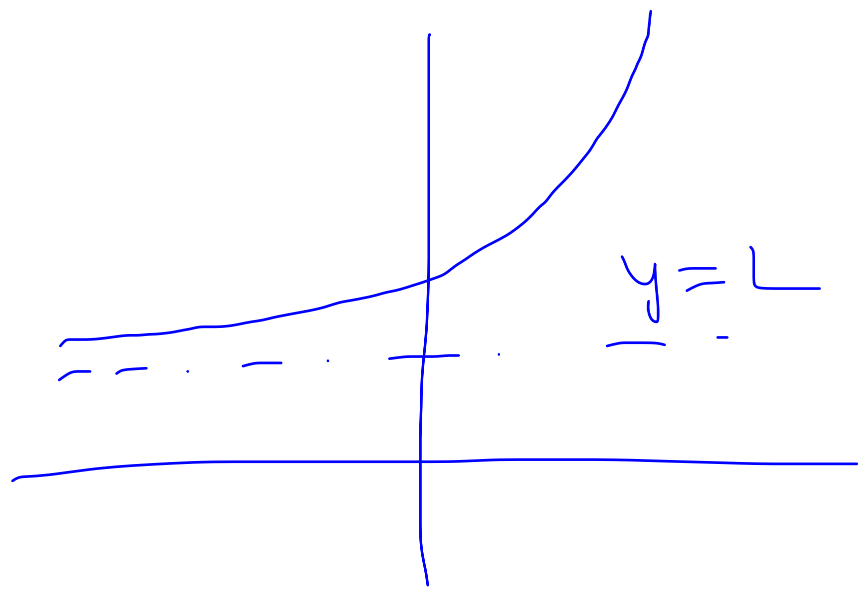
Limits at Infinity

Definition: Let f be defined on (a, ∞) $((-\infty, a))$. Then

$$\lim_{x \to \infty} f(x) = L \qquad \left(\lim_{x \to = -\infty} f(x) = L\right)$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large positive (negative). In other words, $f(x) \to L$ as $x \to \infty$ $(x \to -\infty)$.





y = L is HA

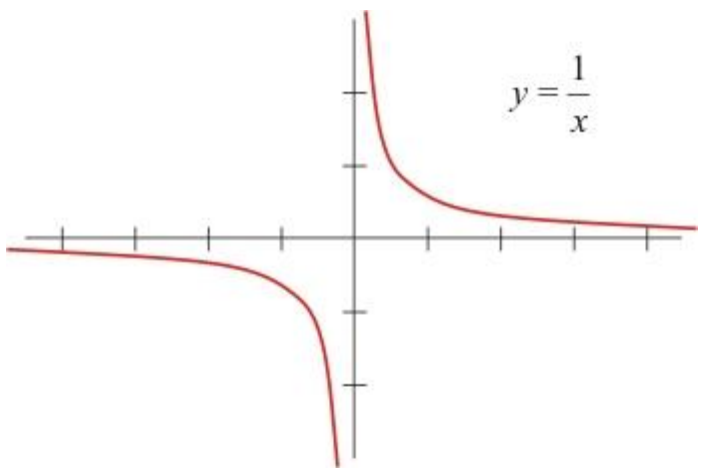
Definition: The line $y = L$ is called a **horizontal asymptote** of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad or \qquad \lim_{x\to=-\infty} f(x) = L$$

Example: $f(x) = \frac{1}{x}$

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$



Fact: If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to\pm\infty} \frac{1}{x^r} = 0$$

Example: Evaluate

(a) $\lim_{x\to\infty} \frac{-x^2+2x-1}{2x^2-x+3} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-1+\frac{2}{x}-\frac{1}{x^2}}{2-\frac{1}{x}+\frac{3}{x^2}} = \frac{-1}{2}$

$\frac{2}{x} \to 0$, $\frac{1}{x^2} \to 0$, $\frac{1}{x} \to 0$, $\frac{3}{x^2} \to 0$

↳ divide by the highest power of the denominator

(b) $\lim_{x\to\infty} \frac{x-2}{x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{2}{x^3}}{1} = 0$$

$\frac{1}{x^2} \to 0$, $\frac{2}{x^3} \to 0$

Ex: $\lim_{x\to-\infty} \frac{x^3-x}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$$= \lim_{x\to-\infty} \frac{x-\frac{1}{x}}{1+\frac{1}{x^2}}$$

$\frac{1}{x} \to 0$, $\frac{1}{x^2} \to 0$

$$= \lim_{x\to-\infty} x = -\infty$$

(c) $\lim_{x\to\infty}(\sqrt{x^2+3}-x)$ $\cdot \dfrac{\sqrt{x^2+3}+x}{\sqrt{x^2+3}+x}$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+3-\cancel{x^2}}{\sqrt{x^2+3}+x}$$

$$= \lim_{x\to\infty} \frac{3}{\sqrt{x^2+3}+x} = 0$$

Ex. $\lim_{x\to\infty} \sqrt{x^2+3x}-x \quad \dfrac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x}$

$$= \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3x}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{3x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{3x}{x^2}}+\frac{x}{x}}$$

$$= \lim_{x\to\infty} \frac{3}{\sqrt{1+\frac{3}{x}}+1} = \frac{3}{\sqrt{1}+1} = \frac{3}{2}$$

$\frac{3}{x} \to 0$

(d) $\lim_{x\to-\infty}(x+\sqrt{x^2+2x})$ $\qquad \dfrac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}$

$$= \lim_{x\to-\infty} \frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty} \frac{-2x}{x-\sqrt{x^2+2x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$\sqrt{x^2} = |x| = -x$$

$$x = -\sqrt{x^2}$$

So $\dfrac{x}{x} - \dfrac{1}{x}\sqrt{x^2+2x}$

$$= 1 - \frac{1}{-\sqrt{x^2}}\sqrt{x^2+2x}$$

$$= 1 + \sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{-2}{1+\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}}} = \lim_{x\to-\infty} \frac{-2}{1+\sqrt{1+\underbrace{\frac{2}{x}}_{\to 0}}}$$

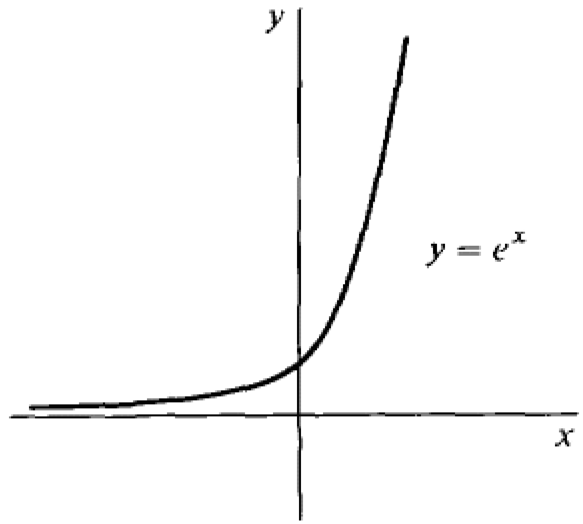
$$= \frac{-2}{1+\sqrt{1}} = -1$$

Example:

$\lim_{x\to-\infty} e^x = 0$

$y=0$ is HA

$$\lim_{x\to\infty} e^x = \infty$$

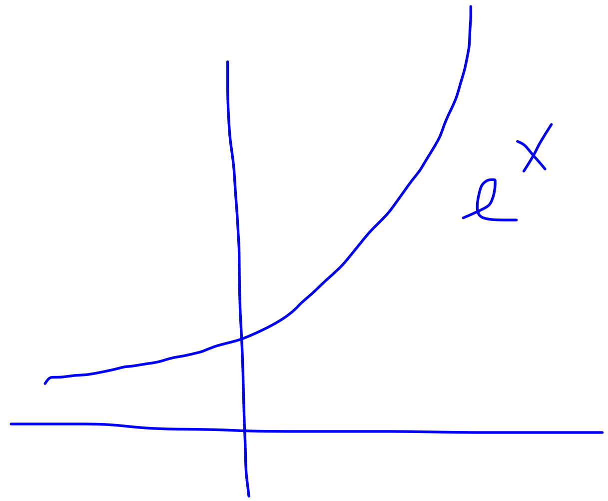


$$\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{t\to-\infty} e^t = 0$$

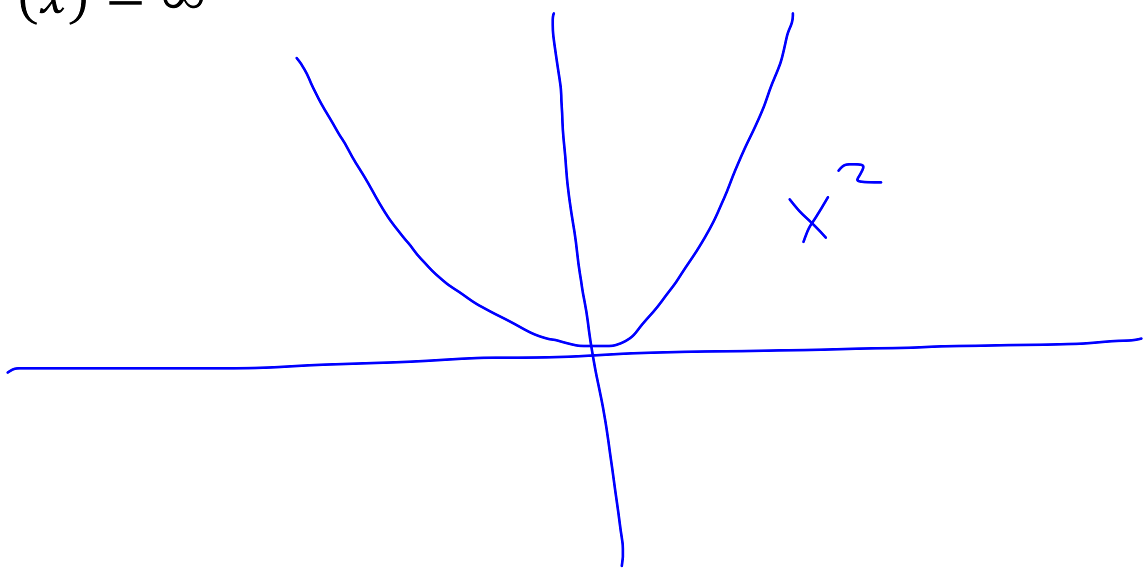
$$t = \frac{1}{x} \xrightarrow[x\to 0^-]{} -\infty$$

Infinite Limits at Infinity

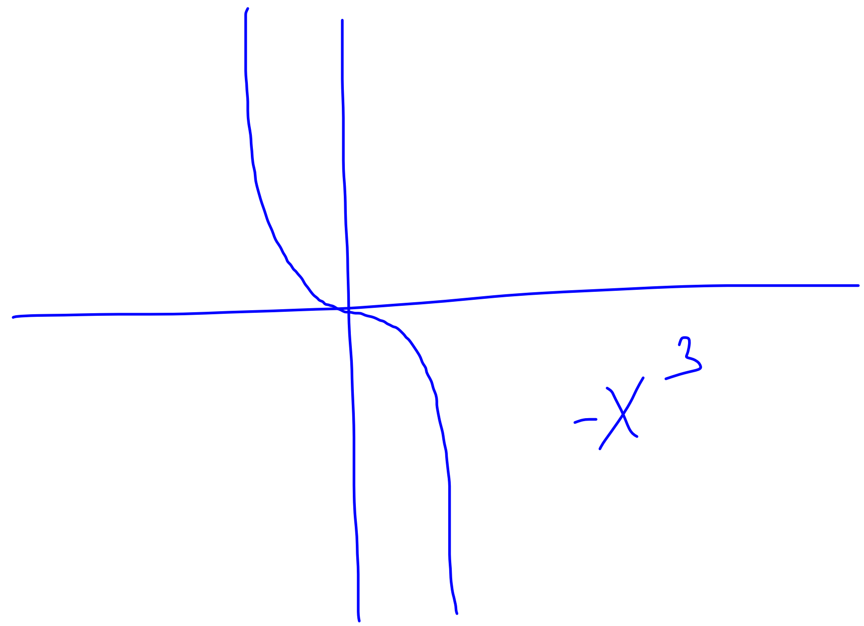
$\lim_{x\to\infty} f(x) = \infty$



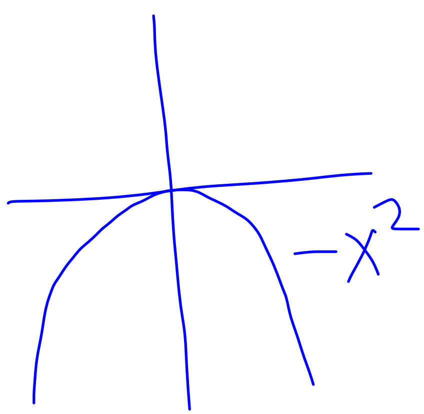
$\lim_{x\to-\infty} f(x) = \infty$

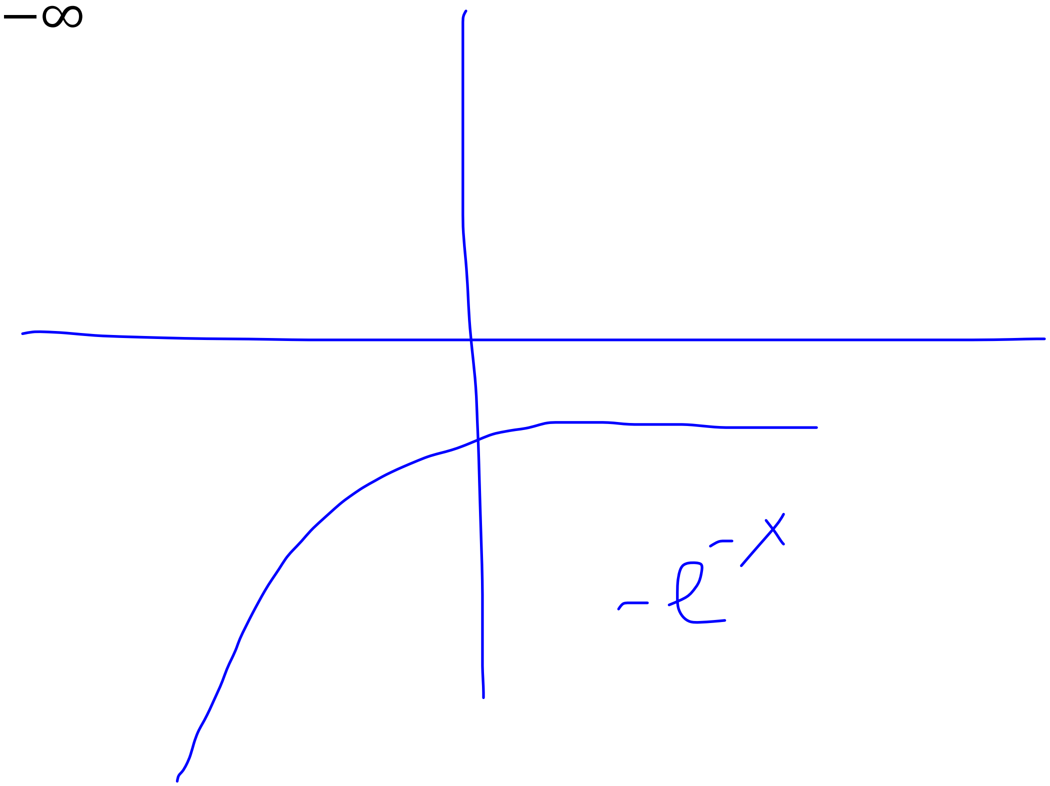


$\lim_{x\to\infty} f(x) = -\infty$



$\lim_{x\to-\infty} f(x) = -\infty$





Example: Evaluate $\lim_{x\to\infty}(x - x^3)$

wrong $\rightarrow$ $\neq \infty - \infty \neq 0$

$$= \lim_{x\to\infty} x(1-x^2) = \lim_{x\to\infty} \underbrace{x}_{\infty} \underbrace{(1-x)}_{-\infty} \underbrace{(1+x)}_{\infty} = -\infty$$

Example: Evaluate $\lim_{x\to-\infty}\frac{x^2-1}{1-x}$ $= \lim_{x\to-\infty} \frac{\cancel{(x-1)}(x+1)}{\cancel{1-x}}$

$$= \lim_{x\to-\infty} -(x+1) = \infty$$

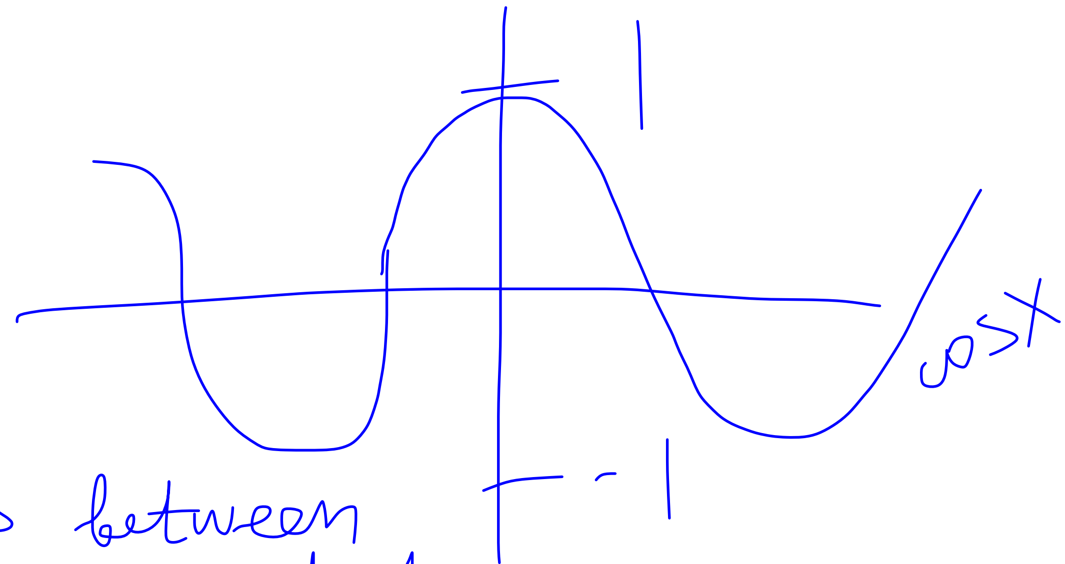
or $\lim_{x\to-\infty} \frac{x^2-1}{1-x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{x - \overbrace{\frac{1}{x}}^{\to 0}}{\underbrace{\frac{1}{x}}_{\to 0} - 1} = \lim_{x\to-\infty} \frac{x}{-1}$

$= \infty$

More examples:

(a) $\lim_{x\to\infty} \cos x$

dne



as cos x oscillates between 1 and -1

(b) $\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}}$

$$= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \lim_{x\to\infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}}$$

$\frac{1}{e^{2x}} \to 0$

$e^{2x} \xrightarrow[x\to\infty]{} \infty$

$$= \frac{1}{1} = 1$$

(c) Prove that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

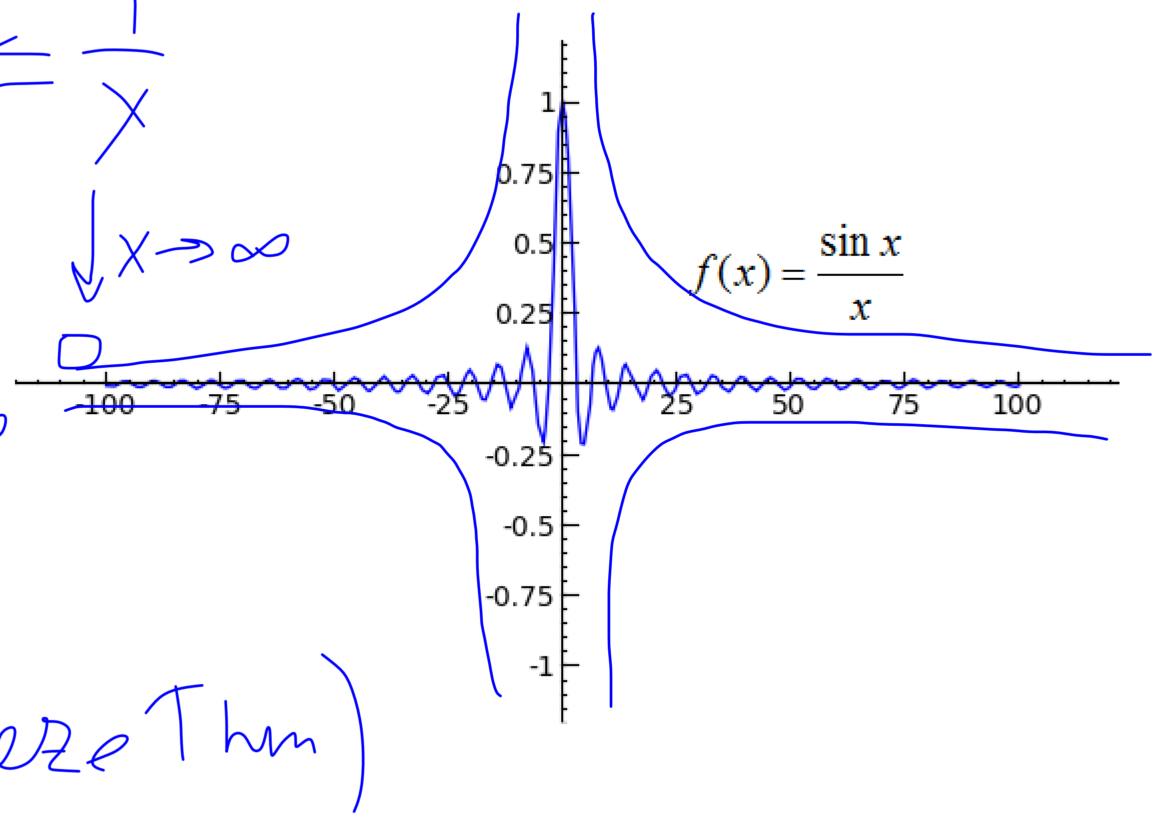
$$-1 \le \sin x \le 1 \quad \cdot \frac{1}{x}$$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\downarrow x\to\infty \qquad \downarrow x\to\infty \qquad \downarrow x\to\infty$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

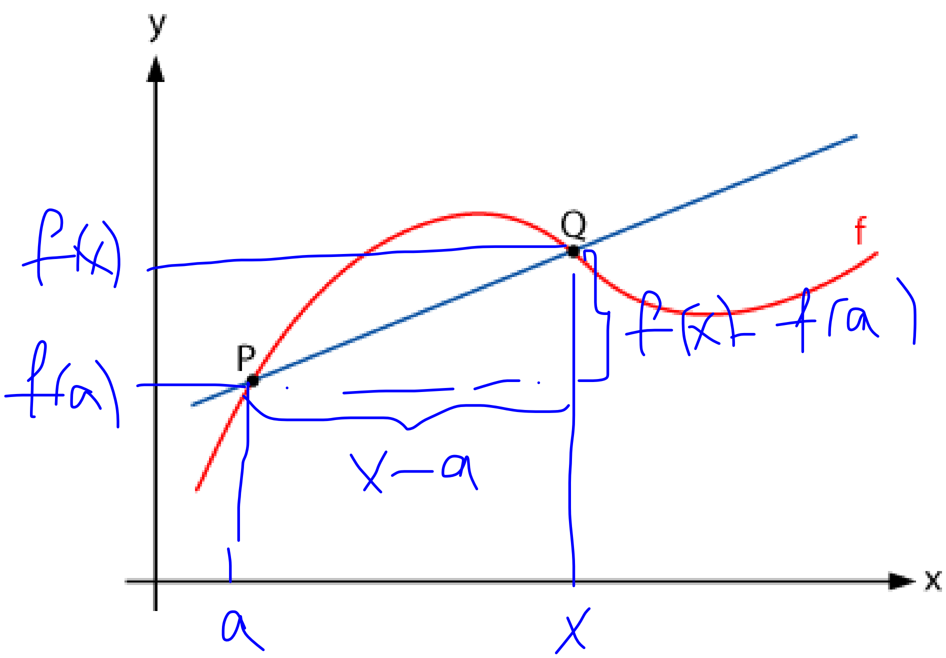
(by the Squeeze Thm)



Derivatives

Recall: The slope of the tangent line, m, is the limit of the slopes of the secant lines:

$$\lim_{Q \to P} m_{PQ} = m$$



$P = (a, f(a))$

$Q = (x, f(x))$

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Slope of tangent line at $P(a, f(a))$

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

Example: Find an equation of the tangent line to $y = x^2 - 2x + 1$ at $P(0,1)$.

$$m = \lim_{x \to 0} \frac{x^2 - 2x + \cancel{1} - \cancel{1}}{x - 0} = \lim_{x \to 0} \frac{x^2 - 2x}{x}$$

$$= \lim_{x \to 0} (x - 2) = -2$$

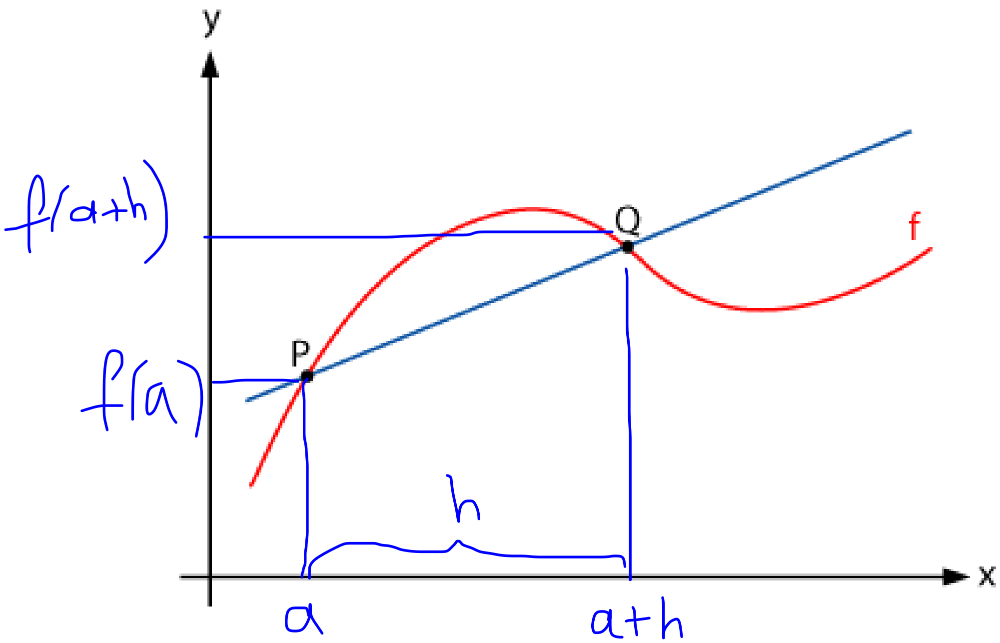
$$y = f(a) + m(x - a)$$

$$y = 1 - 2(x - 0)$$

$$\boxed{y = -2x + 1}$$

Here is another expression for the slope of the tangent line:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



$$x - a = h$$

$$x = a + h$$

$$h = x - a \to 0$$

$$x \to a$$

$$\frac{f(x) - f(a)}{x - a}$$

$$= \frac{f(a+h) - f(a)}{h}$$

Example: Find the slope of the tangent line to $y = \sqrt{x}$ at $P(1,1)$.

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + h - \cancel{1}}{h\left(\sqrt{1+h} + 1\right)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{1+h} + 1\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{\sqrt{1+0} + 1}$$

$$= \frac{1}{2}$$

The types of limits shown above occur so often that they were given a special name.

Definition: The **derivative of a function $\boldsymbol{f}$ at $\boldsymbol{a}$**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

Another notation: $f'(a) = \frac{df}{dx}\Big|_{x=a}$

So the equation of the tangent line to $y = f(x)$ at $(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a)$$

Example: Find the slope of the tangent line to $f(x) = x^2 - 3x + 2$ at $(a, f(a))$.

$$m = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^2 - 3x + \cancel{2} - a^2 + 3a - \cancel{2}}{x - a}$$

$$= \lim_{x \to a} \frac{(x^2 - a^2) - 3(x - a)}{x - a} = \lim_{x \to a} (x + a - 3)$$

$$= a + a - 3 = \boxed{2a - 3}$$

$$m = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} =$$

$$= \lim_{h \to 0} \frac{(a+h)^2 - 3(a+h) + \cancel{2} - a^2 + 3a - \cancel{2}}{h}$$

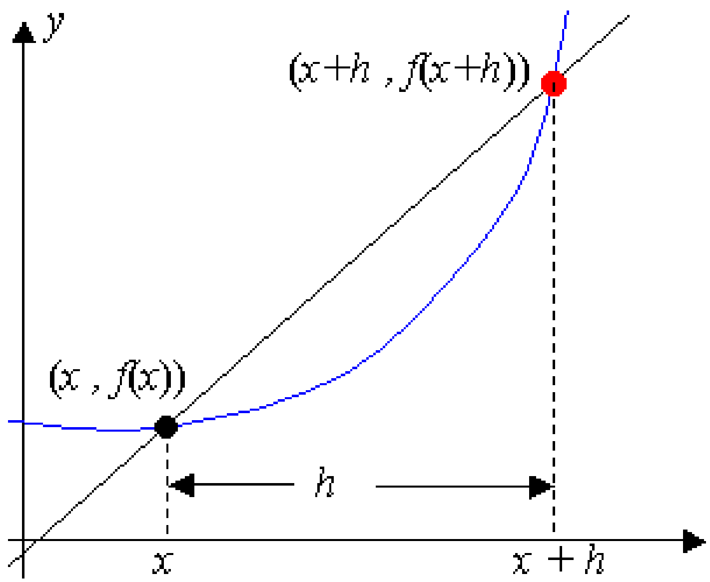
$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{3a} - 3h - \cancel{a^2} + \cancel{3a}}{h} = \lim_{h \to 0} \frac{2ah + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} (2a + h - 3) = \boxed{2a - 3}$$

Let the number *a* vary, i.e. let's replace it with *x*:

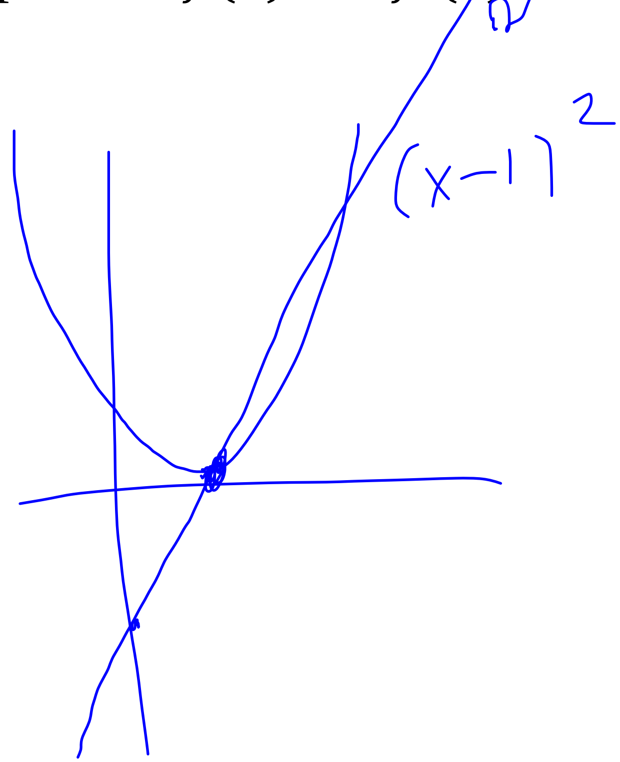
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If the limit exists we regard $f'(x)$ as a new function, called the **derivative of *f*.**



Example: Given $f(x) = x^2 - 2x + 1$, find $f'(x)$. Graph both, $f(x)$ and $f'(x)$.

$$= (x-1)^2$$



$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) + \cancel{1} - x^2 + 2x - \cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h\to 0} (2x + h - 2)$$

$$= 2x - 2 = 2(x-1)$$

Definition: We say f is **differentiable at *a*** if $f'(a)$ exists. It is differentiable on (a, b) if it is differentiable at every number in (a, b).

Theorem: If f is differentiable at a, then f is continuous at a.

Proof: Want : $\lim_{x \to a} f(x) = f(a)$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists}$$

$$\lim_{x \to a} [f(x) - f(a)] \cdot \frac{x-a}{x-a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a) = f'(a) \cdot 0 = 0$$

$$\lim_{x \to a} f(x) = \lim [f(x) - f(a) + f(a)]$$

$$= \lim [f(x) - f(a)] + \lim f(a)$$

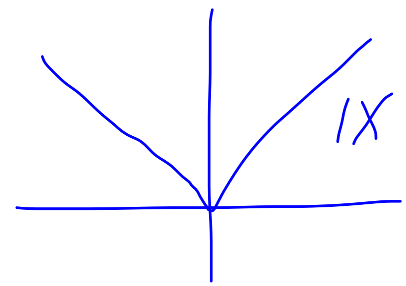
$$= 0 + \lim_{x \to a} f(a) = f(a)$$

∎

Note: The converse of the above theorem is NOT true!

Example: $f(x) = |x|$

$|x|$ is continuous on $(-\infty, \infty)$



$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \text{ dne}$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = -1$$

$$\neq$$

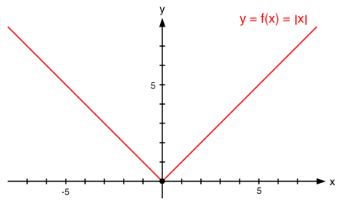
$$\lim_{h \to 0^+} \frac{|h|}{h} = 1$$

$f(x)$ is NOT diff. at $x = 0$.

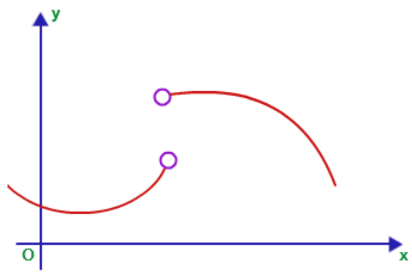
When does function fail to be differentiable?

There are three possibilities:

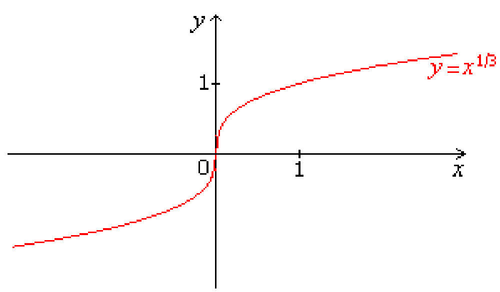
- corner



- discontinuity



- vertical tangent line



Higher Derivatives

Note: if f is differentiable then f' is also a function and might have its own derivative:

$$[f'(x)]' = f''(x)$$

called the **second derivative of f**.

Another notation: $\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2}$.

Similarly,

$$[f''(x)]' = f'''(x)$$

is the **third derivative** of f.

$$[f'''(x)]' = f^{(4)}(x)$$

is the **fourth derivative** of f.

$f^{(n)}(x)$ is the $\boldsymbol{n^{th}}$ **derivative** of f $(\frac{d^n f}{dx^n})$.

Example: Given $f(x) = x^2 - 2x + 1$, find $f''(x)$.

$$f'(x) = 2x - 2$$

$$f''(x) = [f'(x)]'$$

$$= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h) - \cancel{2} - 2x + \cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x} + 2h - \cancel{2x}}{h} = 2$$