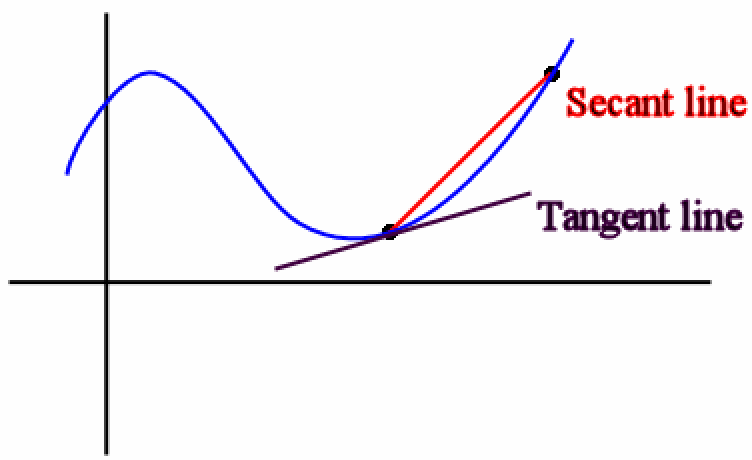
Lecture 2 (Limits)

We shall start with the tangent line problem.

Definition: A **tangent line** (Latin word 'touching') to the function *f(x)* at the point $x = a$ is a line that touches the graph of the function at that point.

A **secant line** (Latin word 'cutting') is a line that cuts a curve more than once.



Example[1]: Let's find a tangent line ($y = mx + b$) to the parabola $f(x) = 15 - 2x^2$ at $x = 1$

$P(1, 13)$

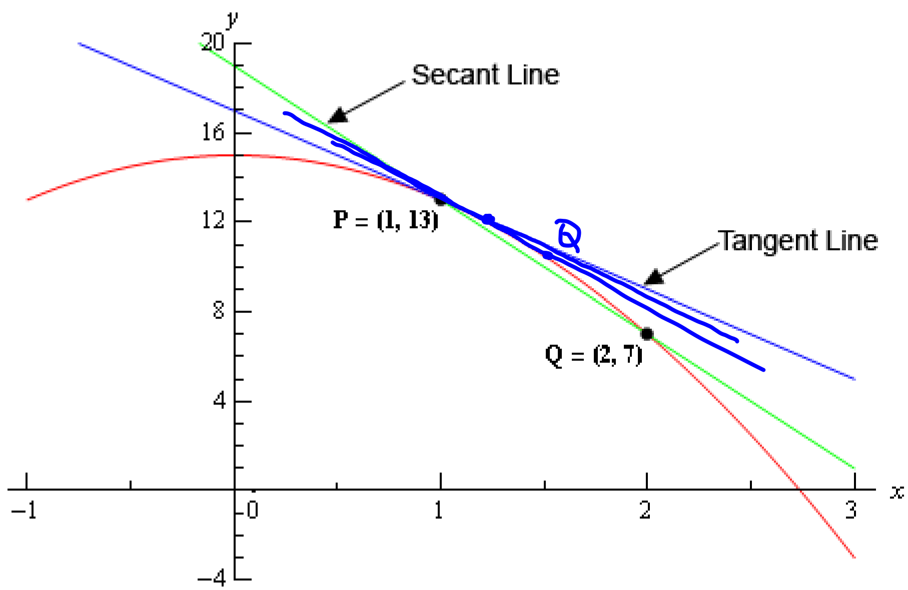
$Q(2, 7)$

$$m_{PQ} = \frac{f(2) - f(1)}{2 - 1} = \frac{7 - 13}{2 - 1} = -6$$

↘ slope of secant line PQ

We can use it as an estimate of m.

But we can do better than that



Take Q closer to P

Consider $Q(x, f(x))$, $m_{PQ} \to m$ as $Q \to P$ i.e. $x \to 1$

[1] Source: http://tutorial.math.lamar.edu/Classes/CalcI/Tangents_Rates.aspx

We can get a formula by finding the slope between P and Q using the general form of $Q = (x, f(x))$, i.e.

$$m_{PQ} = \frac{f(x) - f(1)}{x - 1} = \frac{15 - 2x^2 - 13}{x - 1} = \frac{2 - 2x^2}{x - 1} = \frac{-2\cancel{(x-1)}(x+1)}{\cancel{x-1}} = -2(x+1)$$

Now we pick some values of x getting closer and closer to $x = 1$, plug in and get slopes for secant lines:

from the right ↓

x	m_{PQ}	x	m_{PQ}
2	-6	0	-2
1.5	-5	0.5	-3
1.1	-4.2	0.9	-3.8
1.01	-4.02	0.99	-3.98
1.001	-4.002	0.999	-3.998
1.0001	-4.0002	0.9999	-3.9998

↓ from the left

↓ -4 ↓ -4

We say that the slope of the tangent line, m, is the **limit** of the slopes of the secant lines:

$$\lim_{Q \to P} m_{PQ} = m = -4$$

The equation of the line that goes through $(a, f(a))$ is given by

$$y = f(a) + m(x - a)$$

Thus, the equation of the tangent line to $f(x) = 15 - 2x^2$ at $x = 1$ is

$m = -4$, $a = 1$, $f(a) = 13$

$$y = 13 - 4(x - 1)$$

$$= 13 - 4x + 4$$

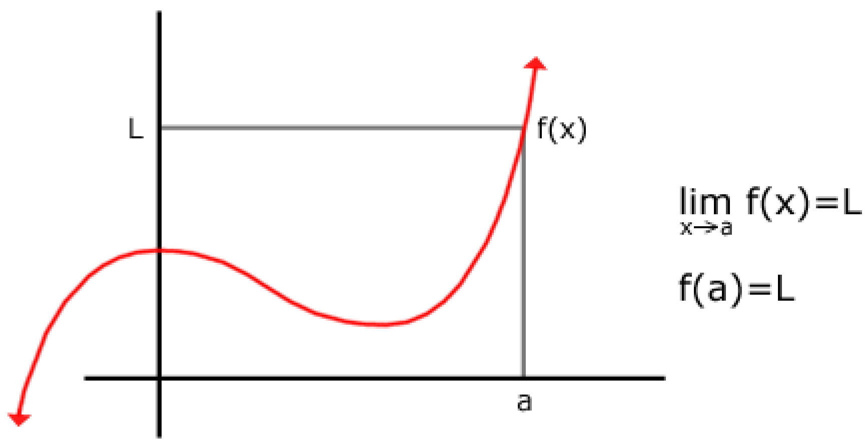
$$\boxed{y = -4x + 17}$$

Limit of Function

Definition: Let $f(x)$ be defined on some open interval that contains number a (except possibly at a itself). Then

$$\lim_{x \to a} f(x) = L$$

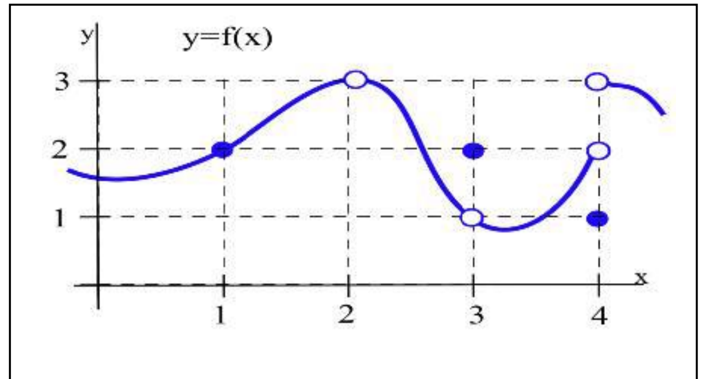
if we can make the values of $f(x)$ arbitratrily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.



Example: Given graph of $y = f(x)$ find the following limits:

(a) $\lim_{x\to 1} f(x)$

$= 2$



(b) $\lim_{x\to 2} f(x)$

$= 3$

although $f(x)$ is not defined at $x=2$

(c) $\lim_{x\to 3} f(x)$

$= 1$

although $f(3) = 2$

(d) $\lim_{x\to 4} f(x)$ does not exsist (dne)

although $f(4) = 1$

Example [2]: Determine the value of $\lim_{x\to 3}\frac{2x^2-x-1}{x-1} = \frac{2\cdot 3^2-3-1}{3-1} = 7$

Construct the table of values:

x	f(x)
2.9	6.82
2.9997	6.9994
2.999993	6.999986
2.9999999	6.9999998
↓	↓
3	**7**

x	f(x)
3.1	7.2
3.004	7.008
3.0001	7.0002
3.000002	7.000004
↓	↓
3	**7**

Thus,

$$\lim_{x\to 3}\frac{2x^2-x-1}{x-1} = 7$$

[2] http://www.saylor.org/site/wp-content/uploads/2011/11/2-2FunctionLimit.pdf

Example (continued): $\lim_{x\to 1}\frac{2x^2-x-1}{x-1}$ $\leftarrow \frac{0}{0}$

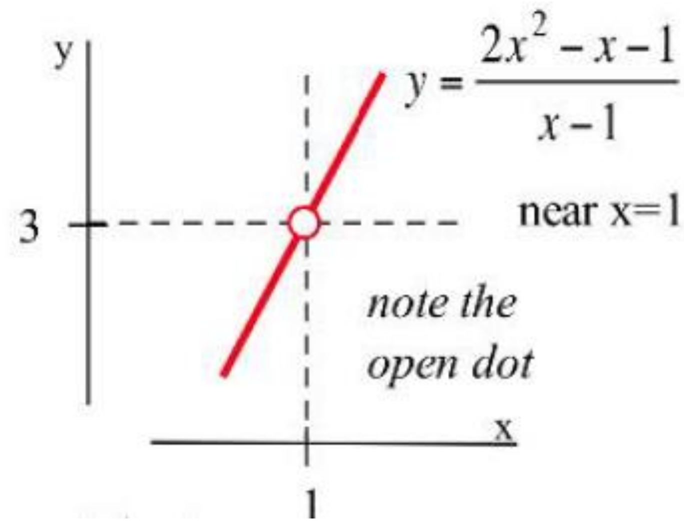
$$\frac{2x^2-x-1}{x-1} = \frac{(2x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

$$= 2x+1$$

$$\lim_{x\to 1}\frac{2x^2-x-1}{x-1} = \lim_{x\to 1}(2x+1)$$

$$= 2\cdot 1+1 = 3$$

$\hookrightarrow$ (applying limit laws to be defined later)



x	f(x)
0.9	2.82
0.9998	2.9996
0.999994	2.999988
0.9999999	2.9999998
↓	↓
1	**3**

x	f(x)
1.1	3.2
1.003	3.006
1.0001	3.0002
1.000007	3.000014
↓	↓
1	**3**

Example: What can you say about $\lim_{x\to 0} \sin\frac{\pi}{x}$?

$f(x) = \sin\frac{\pi}{x}$ ↳ not defined at 0

$f(1) = \sin\pi = 0$

$f(\frac{1}{2}) = \sin 2\pi = 0$

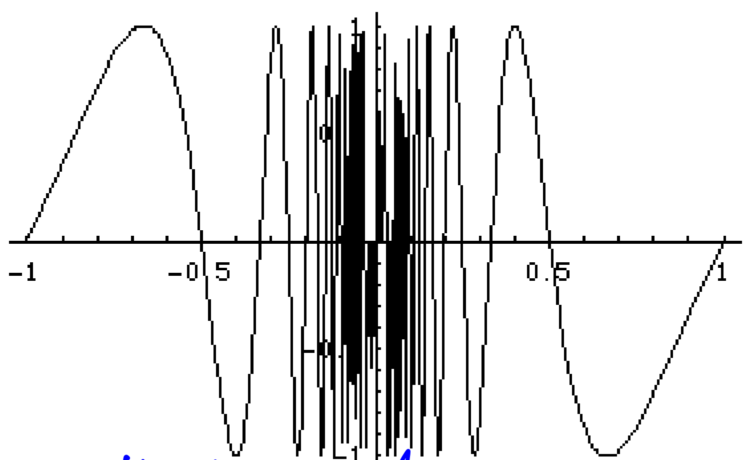
$f(\frac{1}{4}) = \sin 4\pi = 0$

$f(\frac{1}{100}) = \sin 100\pi = 0$

$\vdots$

So, is limit zero?

No!

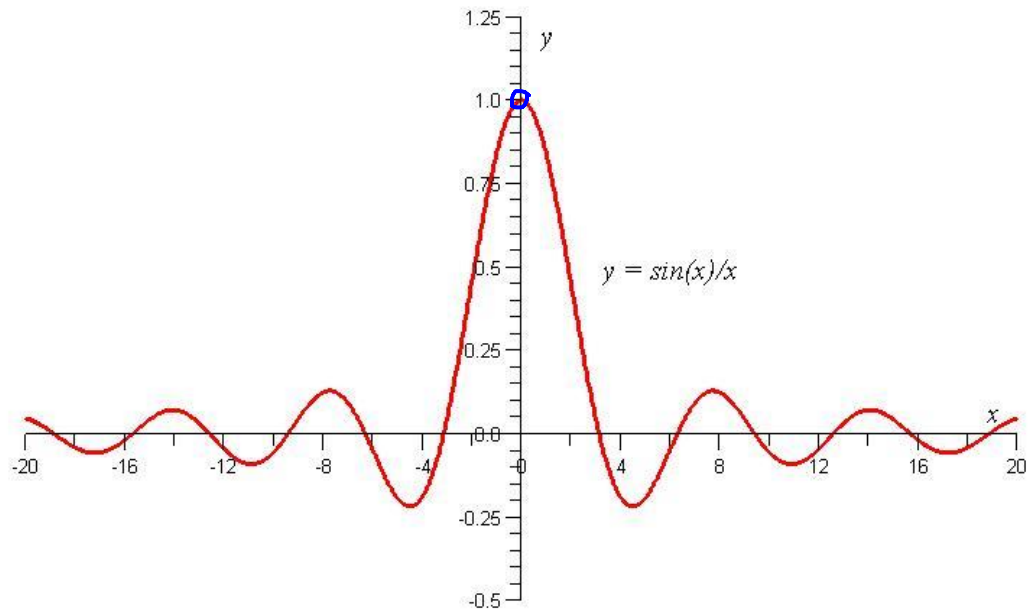


Values of $f(x)$ oscillate between 1 and -1 infinitely often. Thus, limit dne.

Example (famous limit): $\lim_{x \to 0} \frac{\sin x}{x}$

Table of values:

x (radians)	-0.7	-0.2	-0.05	0	0.01	0.03	0.3	1.4
$\sin x / x$	0.92031	0.993347	0.999583	***	0.999983	0.99985	0.98506	0.703893



$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

One-Sided Limits

<u>Definition</u>: We write

$$\lim_{x \to a^-} f(x) = L$$

$x < a$

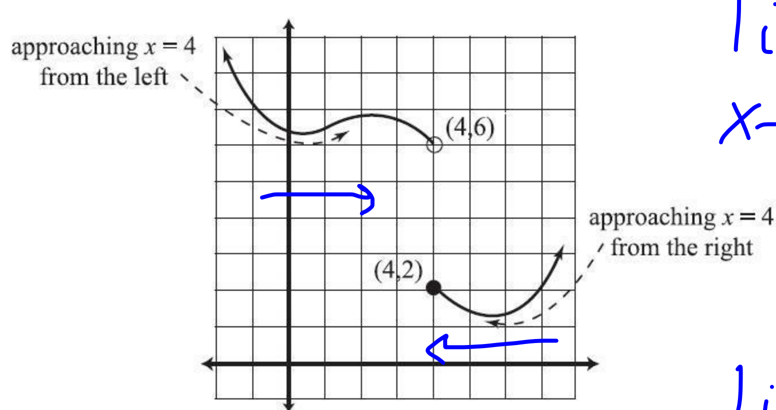
to denote the limit of $f(x)$ as x approaches a from the left (**left-hand limit**).

We write

$$\lim_{x \to a^+} f(x) = L$$

$x > a$

to denote the limit of $f(x)$ as x approaches a from the right (**right-hand limit**).



$\lim_{x \to 4^-} f(x) = 6$

$\neq$

$\lim_{x \to 4^+} f(x) = 2$

$\lim_{x \to 4} f(x)$ dne

Theorem: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$

Corollary: If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x) = L$ does not exist.

Example: Let $f(x) = \begin{cases} 1, & x < 1 \\ x, & 1 \leq x \leq 3 \\ 2, & 3 < x \end{cases}$.

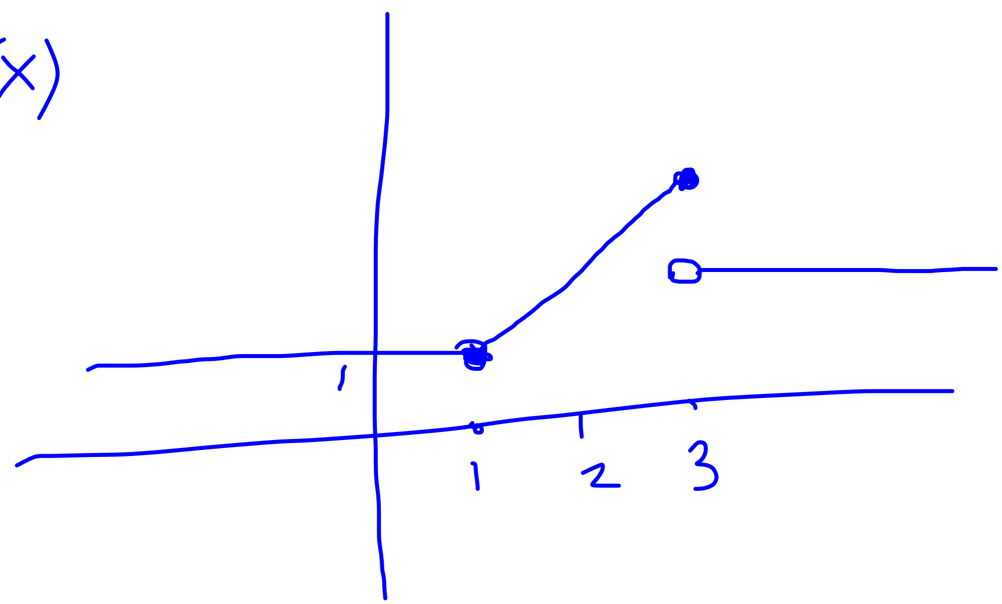
Find the one and two-sided limits of f at 1 and 3.

$$\lim_{x\to 1^-} f(x) = 1 = \lim_{x\to 1^+} f(x)$$

$$\Rightarrow \lim_{x\to 1} f(x) = 1$$

$$\lim_{x\to 3^-} f(x) = 3 \neq 2 = \lim_{x\to 3^+} f(x)$$

$$\Rightarrow \lim_{x\to 3} f(x) \text{ dne}$$



Infinite Limits $\pm\infty$

Consider $\lim_{x\to 0} \frac{1}{x^2} = \infty$

what about $\lim_{x\to 0} \frac{1}{x}$?

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$

$\lim_{x\to 0^+} \frac{1}{x} = \infty$

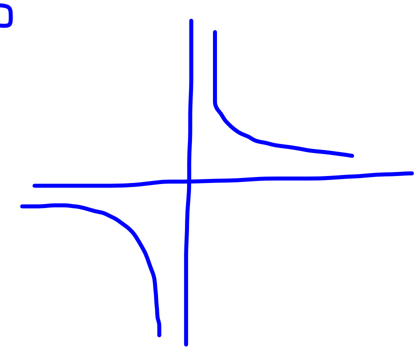
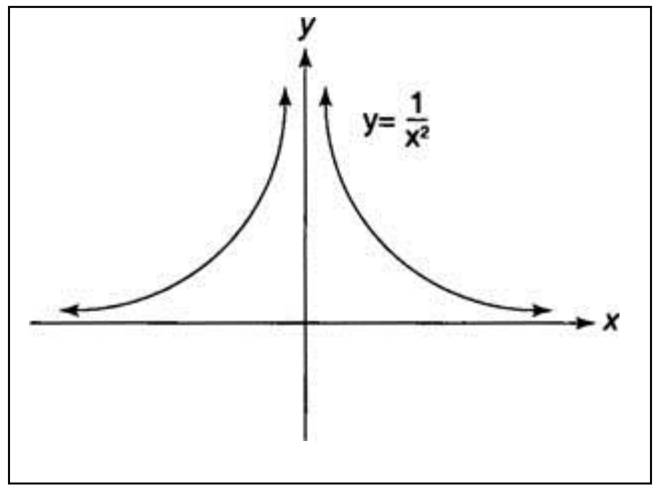


Table of values:

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000



Definition: Let f be a function defined on both sides of a, except possible at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

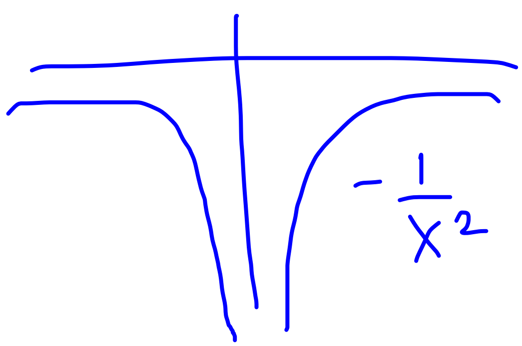
Another notation: $f(x) \to \infty$ as $x \to a$

Similarly, let f be a function defined on both sides of a, except possible at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

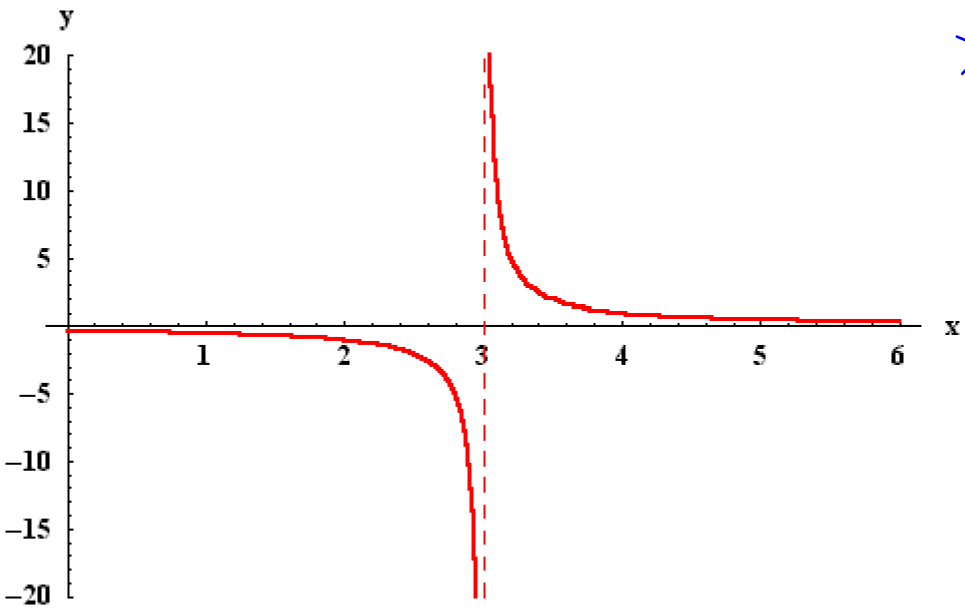
means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Example: $\lim_{x \to 0}(-\frac{1}{x^2}) = -\infty$



Example: Consider function $f(x) = \frac{1}{x-3}$.

$$\lim_{x \to 3^-} \frac{1}{x-3} = -\infty$$



$$\lim_{x \to 3^+} \frac{1}{x-3} = \infty$$

$x = 3$ is VA

Note: When x gets closer to 3, then the points on the graph get closer to the (dashed) vertical line $x = 3$. Such a line is called a **vertical asymptote**. (VA)

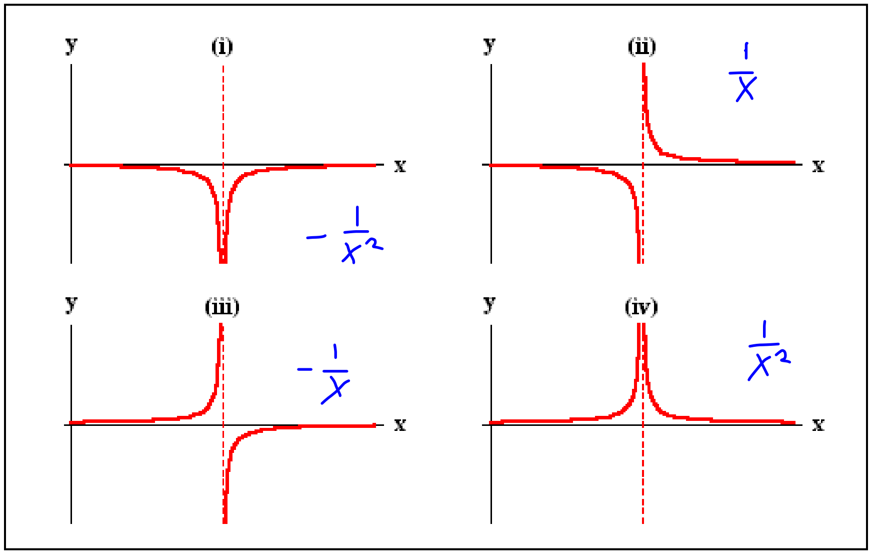
For a given function $f(x)$, there are four cases, in which vertical asymptotes can present themselves:

(i) $\lim_{x\to a^-} f(x) = -\infty$; $\lim_{x\to a^+} f(x) = -\infty$;

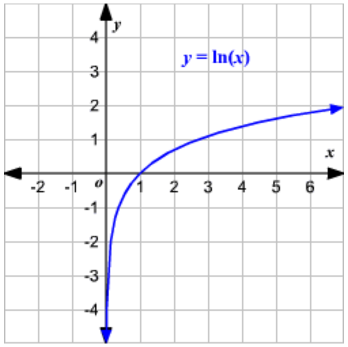
(ii) $\lim_{x\to a^-} f(x) = -\infty$; $\lim_{x\to a^+} f(x) = +\infty$;

(iii) $\lim_{x\to a^-} f(x) = +\infty$; $\lim_{x\to a^+} f(x) = -\infty$;

(iv) $\lim_{x\to a^-} f(x) = +\infty$; $\lim_{x\to a^+} f(x) = +\infty$;



<u>Example</u>: $y = \ln x$



$$\lim_{x\to 0^+} \ln x = -\infty$$

$x = 0$ is VA

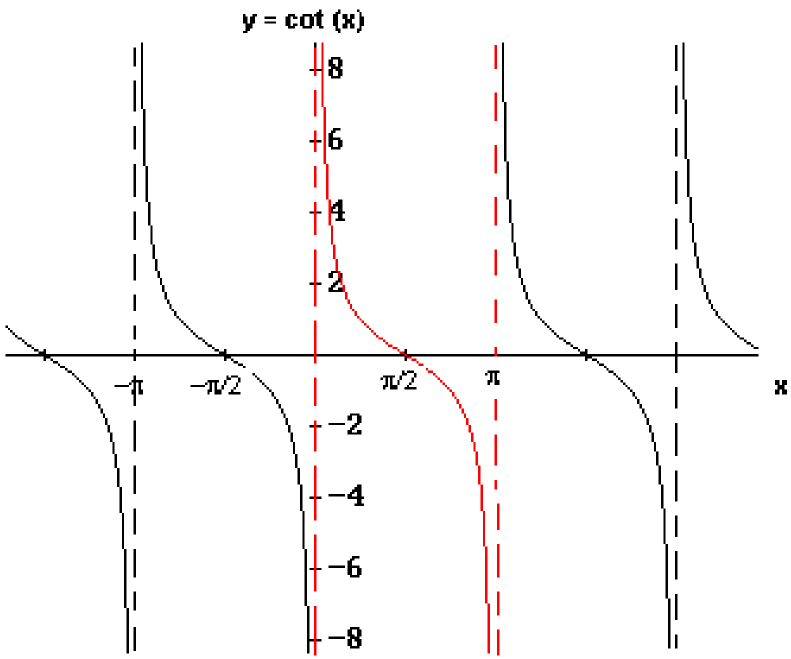
Example: $y = \cot x = \dfrac{\cos x}{\sin x} \xrightarrow[x \to 0^-]{} -\infty$

$\cos x \to 1$

$\sin x \to 0$
< 0

$$\lim_{x \to 0^-} \cot x = -\infty$$

$$\lim_{x \to 0^+} \cot x = \infty$$



$x = 0$ is VA

In general, $x = k\pi$ is VA
$k \in \mathbb{Z}$

Limit Laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Example: Given $\lim_{x\to 1} f(x) = 3$ and $\lim_{x\to 1} g(x) = -2$, find

$$\lim_{x\to 1}[2f(x) - 5g(x)]$$

$$= 2\lim_{x\to 1} f(x) - 5\lim_{x\to 1} g(x)$$

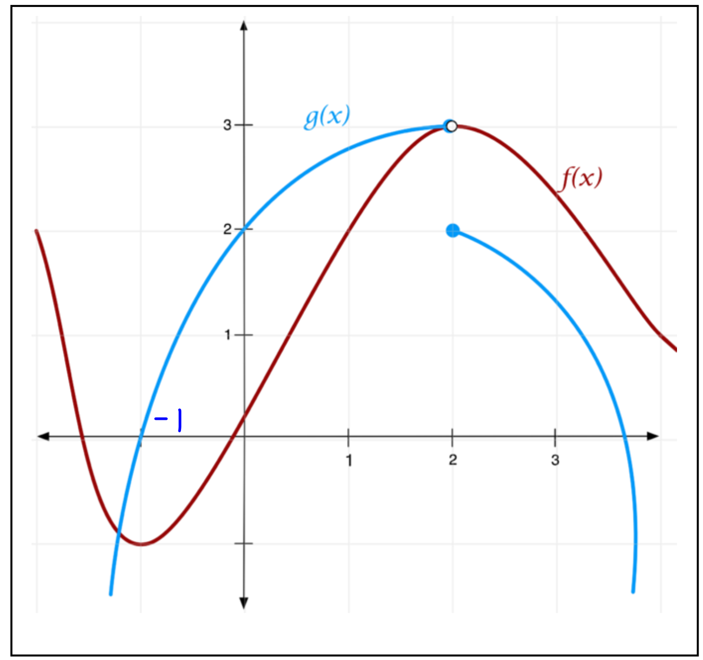
$$= 2\cdot 3 - 5(-2) = 16$$

Example: Use the limit laws and the graphs of f and g to evaluate the following limits, if they exist.

(a) $\lim_{x\to -1}[f(x) - 2g(x)]$

$= \lim_{x\to -1} f(x) - 2\lim_{x\to -1} g(x) = -1 - 2\cdot 0$

$= -1$



(b) $\lim_{x\to -1}[f(x)g(x)]$

$= \lim_{x\to -1} f(x) \lim_{x\to -1} g(x) = -1\cdot 0 = 0$

(c) $\lim_{x\to 2}[f(x)g(x)]$ dne

$= \lim_{x\to 2} f(x) \lim_{x\to 2} g(x)$

3 dne

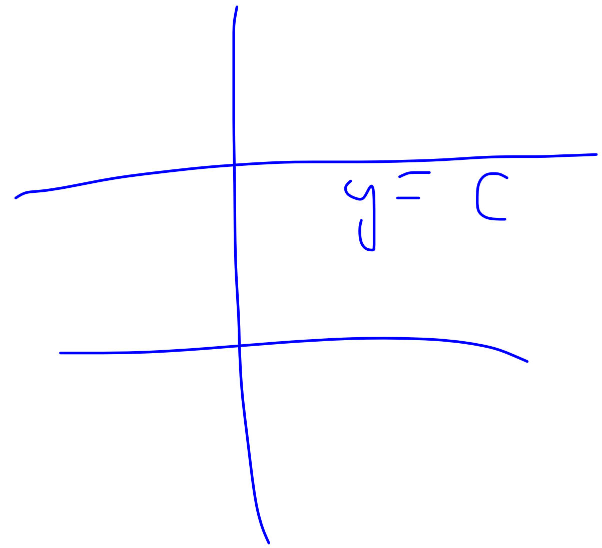
$\lim_{x\to 2} f(x) = 3$

$\lim_{x\to 2^-} g(x) = 3 \neq \lim_{x\to 2^+} g(x) = 2$

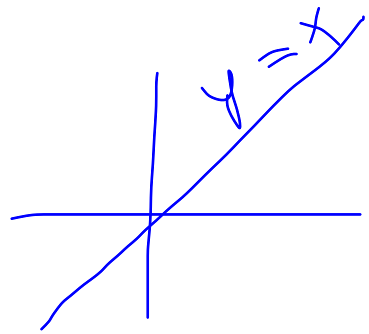
Limit Laws (continued)

6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n,\ n \in \mathbb{Z}^+$

7. $\lim_{x\to a} c = c$



8. $\lim_{x\to a} x = a$



9. $\lim_{x\to a} x^n = a^n,\ n \in \mathbb{Z}^+$

6+8 $\quad = \left(\lim_{x\to a} x\right)^n = a^n$

10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},\ n \in \mathbb{Z}^+$ (if n is even, we assume that $a > 0$)

11. In general: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)},\ n \in \mathbb{Z}^+$

12. If f is a continuous function (to be studied in the next lecture) at a (i.e. polynomial or a rational function defined at a), then

$$\lim_{x \to a} f(x) = f(a)$$

Example: Evaluate $\lim_{x\to 0}(x^2 - 3x + 5)$ $= f(0)$

$= 0^2 - 3 \cdot 0 + 5 = 5$

Example: Find $\lim_{x\to 2} \frac{1+x}{1-x}$ $= f(2) = \frac{1+2}{1-2} = -3$

Note: not all limits can be evaluated by direct substitution.

Example: Find $\lim_{x\to 1} \frac{x^3-1}{x-1}$ $= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = 1^2+1+1$

not defined at $x=1$ $= 3$

$(x^3 - y^3) = (x-y)(x^2 + xy + y^2)$

$+ \quad + \quad -$

Example: $\lim_{x\to 0} \frac{\sqrt{x^2+25}-5}{x^2}$ not defined at $x=0$

$$\frac{\sqrt{x^2+25}-5}{x^2} \cdot \frac{\sqrt{x^2+25}+5}{\sqrt{x^2+25}+5}$$

$$= \frac{\left(\sqrt{(x^2+25)}\right)^2 - 5^2}{x^2\left(\sqrt{x^2+25}+5\right)} = \frac{x^2 + \cancel{25} - \cancel{25}}{x^2\left(\sqrt{x^2+25}+5\right)}$$

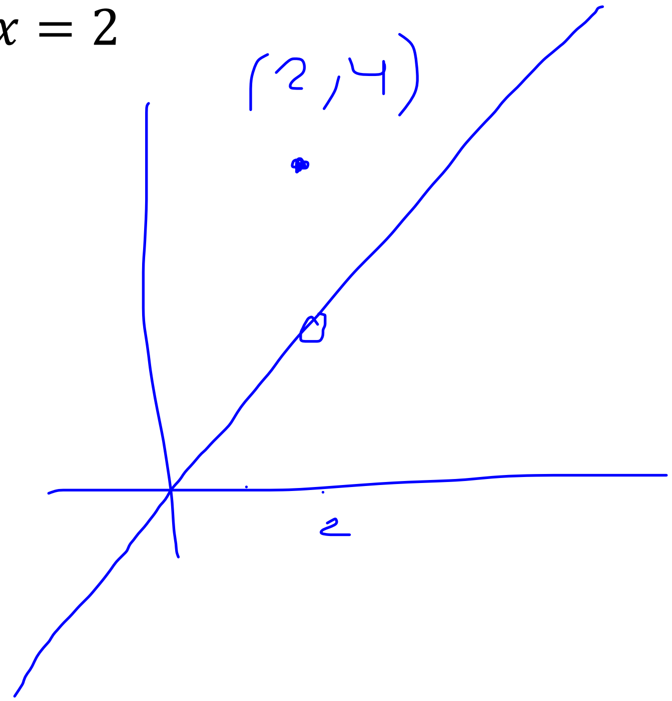
$$= \frac{\cancel{x^2}}{\cancel{x^2}\left(\sqrt{x^2+25}+5\right)} = \frac{1}{\sqrt{x^2+25}+5}$$

$$\lim_{x\to 0} \frac{\sqrt{x^2+25}-5}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+25}+5} = \frac{1}{\sqrt{0^2+25}+5}$$

$$= \frac{1}{10} = 0.1$$

Fact: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ (provided that limits exist).

Example: Find $\lim_{x\to 2} g(x)$ given $g(x) = \begin{cases} x, & x \neq 2 \\ 4, & x = 2 \end{cases}$



Define $f(x) = x$

Then $f(x) = g(x)$, $x \neq 2$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x = 2$$

$\| \to$ fact

$$\lim_{x\to 2} g(x)$$

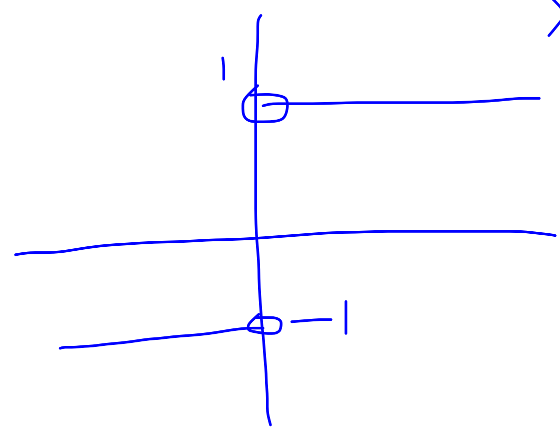
<u>Theorem</u>:

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

$\frac{|x|}{x}$



<u>Example</u>: Investigate $\lim_{x\to 0} \frac{|x|}{x}$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

$$\frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$$

$x < 0$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} (1) = 1$$

$x \to 0^+,\ x > 0$

$$\lim_{x\to 0} \frac{|x|}{x} \text{ dne}$$

Example: If $f(x) = \begin{cases} \sqrt{x-1}, & x > 1 \\ 1-x, & x < 1 \end{cases}$, determine whether $\lim_{x\to 1} f(x)$ exists.

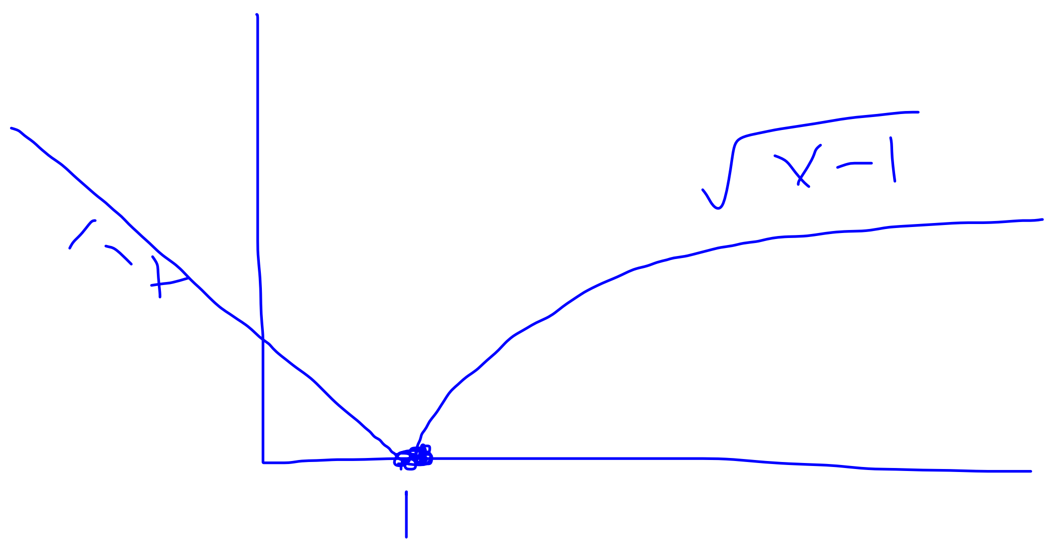
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 1-1 = 0$$

$x < 1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x-1} = \sqrt{1-1} = 0$$

$x > 1$

$$\lim_{x\to 1} f(x) = 0$$

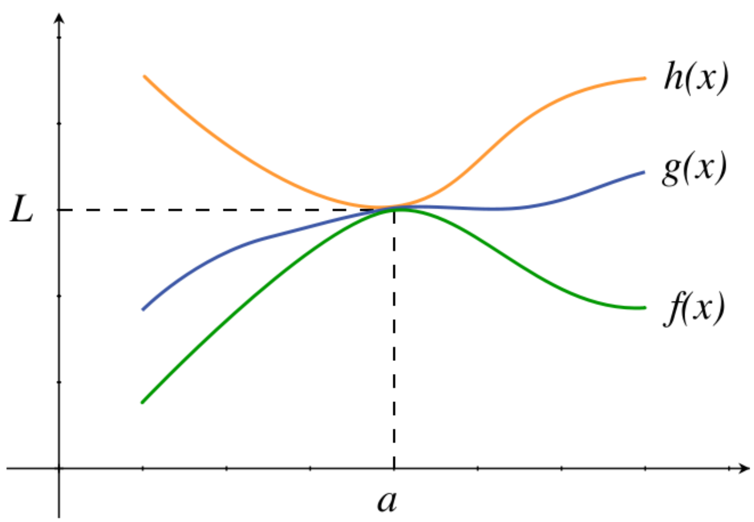


The Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ when x is near a (except possible at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$



Theorem of two policemen

$$f(x) \le g(x) \le h(x)$$

Example: Show that $\lim_{x\to 0} \underbrace{x^2 \sin\frac{1}{x}}_{g(x)} = 0$

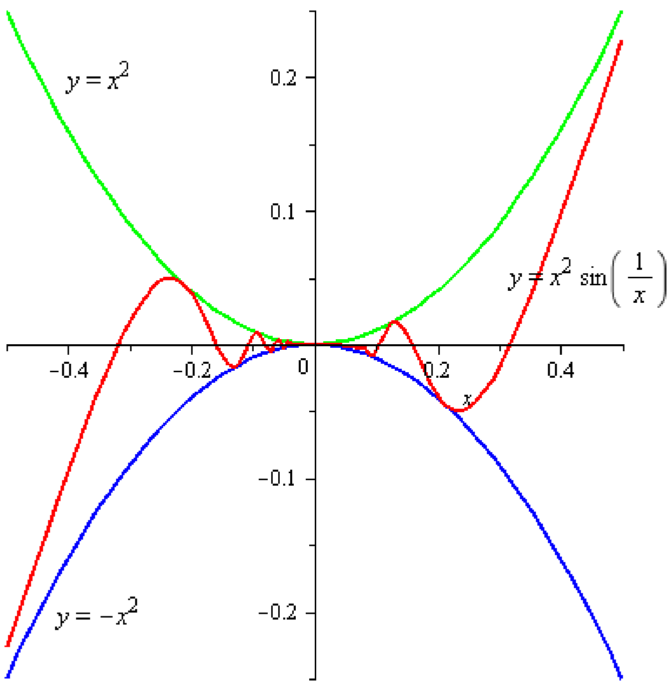
$-1 \le \sin\frac{1}{x} \le 1 \quad \cdot \; x^2 \ge 0$

$\underbrace{-x^2}_{f(x)} \le \underbrace{x^2 \sin\frac{1}{x}}_{g(x)} \le \underbrace{x^2}_{h(x)}$

$f(x) \xrightarrow{x \to 0} 0$, $\quad h(x) \xrightarrow{x \to 0} 0$

by Squeeze Thm

$g(x) \xrightarrow{x \to 0} 0$



Precise Definition of Limit

The previous definition of a limit that we used is quite intuitive. We say 'as x is close to a, $f(x)$ gets closer and closer to L'. What does this mean?

Example: $f(x) = 2x + 3$. What happens if x gets close to 2?

Intuitively we know that $\lim_{x\to 2}(2x + 3) = 7$.

What if we want to know how close x must be to 2 so that $f(x)$ differs from 7 by less than 0.1?

We denote by $|x - 2|$ the distance from x to 2 and $|f(x) - 7|$ the distance from $f(x)$ to 7.

Problem: Find a number $\delta > 0$ such that $|f(x) - 7| < 0.1$ if $|x - 2| < \delta$.

$f(x) = 2x+3$

$|f(x) - 7| < 0.1$

$|2x+3-7| < 0.1$

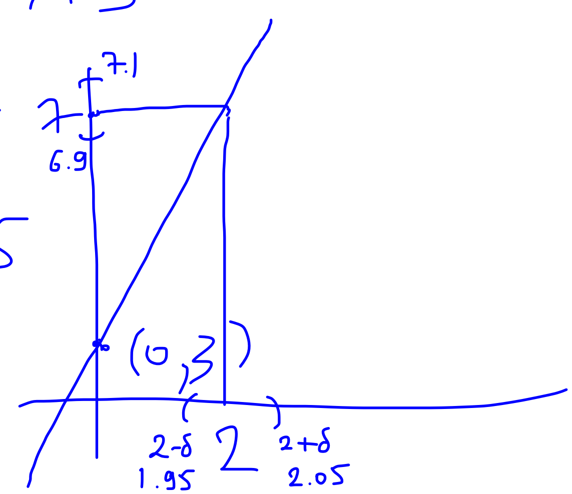
$|2x-4| < 0.1$

$|2| \cdot |x-2| < 0.1$

$2|x-2| < 0.1$

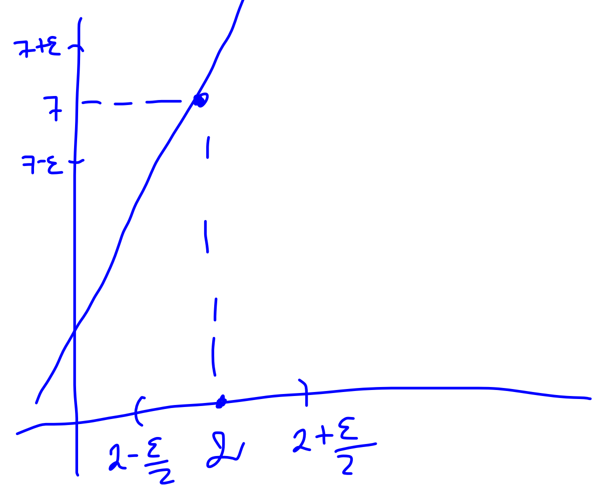
$|x-2| < 0.05$

Choose $\delta = 0.05$



Problem: Find a number $\delta > 0$ such that $|f(x) - 7| < 0.01$ if $|x - 2| < \delta$.

$$|f(x) - 7| < 0.01$$
$$|2x - 3 - 7| < 0.01$$
$$|x - 2| < \frac{0.01}{2} = 0.005$$
Choose $\delta = 0.005$



Problem: Find a number $\delta > 0$ such that $|f(x) - 7| < \varepsilon$ if $|x - 2| < \delta$, for any arbitrarily small $\varepsilon > 0$.

$$|f(x) - 7| < \varepsilon$$
$$|2x + 3 - 7| < \varepsilon$$
$$|2x - 4| < \varepsilon$$
$$2|x - 2| < \varepsilon$$
$$|x - 2| < \varepsilon/2$$

Choose $\delta = \dfrac{\varepsilon}{2}$

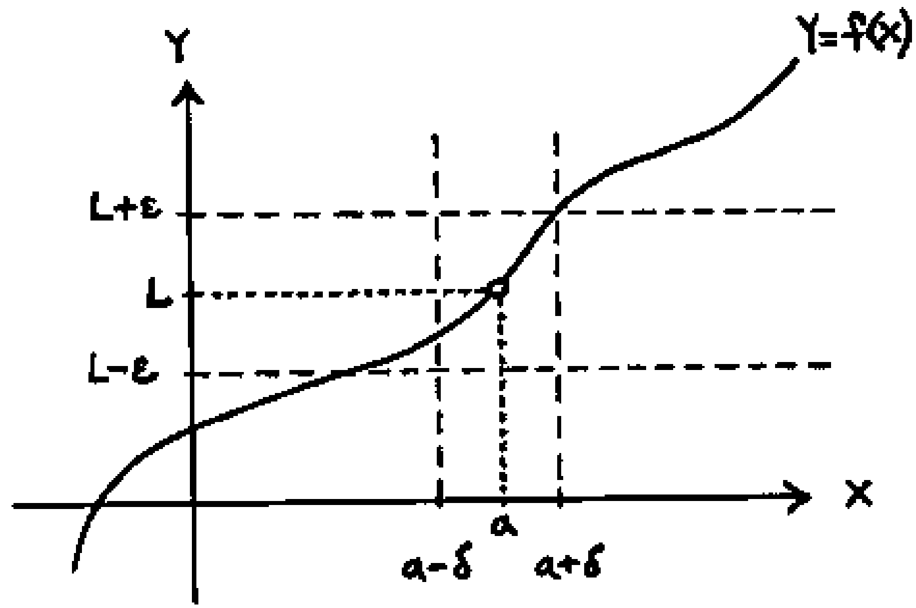
So, we can make the values of $f(x)$ within an arbitrary distance ε from 7 by taking x within distance $\varepsilon/2$ from 2.

Definition: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$



$f(x)$

Example: Use ε, δ definition to show that $\lim_{x\to 3}(7x+2) = 23$.

$\delta(\varepsilon)$

Need to show:

for any $\varepsilon > 0$, there is $\delta > 0$ s.t.

if $0 < |x-3| < \delta \Rightarrow |f(x) - 23| < \varepsilon$

$|f(x) - 23| < \varepsilon$

$|7x + 2 - 23| < \varepsilon$

$|7x - 21| < \varepsilon$

$7|x-3| < \varepsilon$

$|x-3| < \frac{\varepsilon}{7}$

1. Choose $\delta = \frac{\varepsilon}{7}$

repeat steps

2. Show this δ works

$|x-3| < \delta = \frac{\varepsilon}{7}$

$|f(x) - 23| < \varepsilon$

$a - \delta$ $\quad$ a $\quad$ $a + \delta$

L $\quad$ R

Similarly we can define:

- **Left-hand limit**: $\lim_{x \to a^-} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a - \delta < x < a$ then $|f(x) - L| < \varepsilon$

- **Right-hand limit**: $\lim_{x \to a^+} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a < x < a + \delta$ then $|f(x) - L| < \varepsilon$

<u>Example</u>: Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$

for every $\varepsilon > 0$, there is $\delta > 0$ s.t.

$0 < x < 0 + \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon$

$0 < x < \delta$

$|\sqrt{x} - 0| < \varepsilon$

$|\sqrt{x}|^2 < \varepsilon^2$

$x < \varepsilon^2$

Choose $\delta = \varepsilon^2$

Show that $\delta = \varepsilon^2$ works

If $0 < x < \delta$

$\Rightarrow 0 < x < \varepsilon^2$

$\sqrt{x} < \varepsilon$

$|\sqrt{x} - 0| < \varepsilon$

$\Rightarrow \lim_{x \to 0^+} \sqrt{x} = 0$

- **Infinite limits**: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

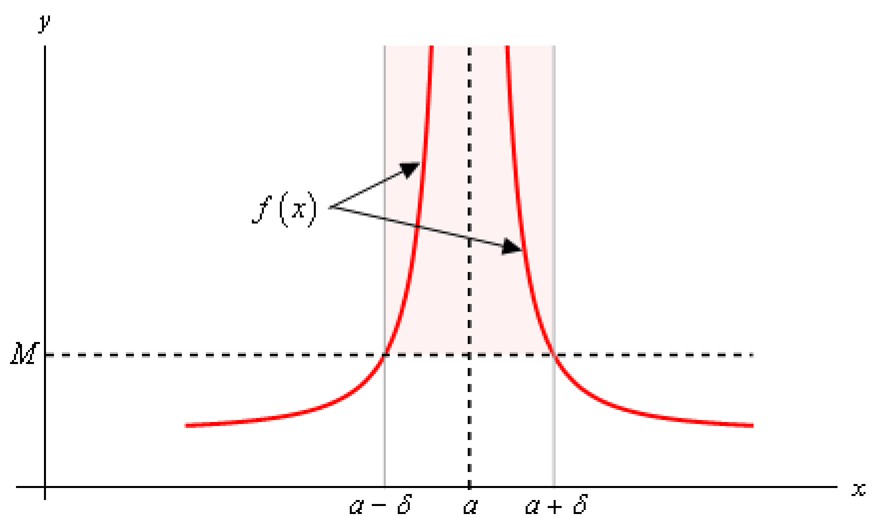
$$\lim_{x \to a} f(x) = +\infty$$

$$\left(\lim_{x \to a} f(x) = -\infty\right)$$

means that for every positive number M (negative number N) there is a positive number δ such that if

$$0 < |x - a| < \delta \text{ then } f(x) > M$$

$$(f(x) < N)$$



<u>Example</u>: Use the definition of the limit to show that

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty$$

Given any $M > 0$, there is $\delta > 0$ s.t.

whenever $0 < |x - 0| < \delta \implies \frac{1}{x^2} > M$

$|x| < \delta$

$$\frac{1}{x^2} > M$$

$$x^2 < \frac{1}{M}$$

$$|x| < \sqrt{\frac{1}{M}} = \frac{1}{\sqrt{M}}$$

Choose $\delta = \frac{1}{\sqrt{M}}$

Show that $\delta = \frac{1}{\sqrt{M}}$ works

If $|x| < \delta = \frac{1}{\sqrt{M}}$

$$x^2 < \frac{1}{M}$$

$$\frac{1}{x^2} > M$$

(for any $M > 0$)

$\implies \lim_{x \to 0} \frac{1}{x^2} = \infty$

Ex.

Dirichlet function

$$D(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$\lim_{x \to 0} D(x) = L \leftarrow$ assume limit exists. Then

for any $\varepsilon > 0$ there is $\delta > 0$ s.t.

$0 < |x - 0| < \delta \Rightarrow |D(x) - L| < \varepsilon$

Choose $\varepsilon = \frac{1}{2}$. $0 < |x| < \delta \Rightarrow |D(x) - L| < \frac{1}{2}$

Take any rational $r \in \mathbb{Q}$ s.t. $0 < |r| < \delta$

Then $D(r) = 0 \Rightarrow |0 - L| < \frac{1}{2} \Rightarrow L \leq |L| < \frac{1}{2}$

Now take any irrational $s \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $0 < |s| < \delta$

Then $D(s) = 1 \Rightarrow |1 - L| < \frac{1}{2} \Rightarrow 1 - L \leq |1 - L| < \frac{1}{2}$

$\Rightarrow L > 1 - \frac{1}{2} \Rightarrow L > \frac{1}{2} \Rightarrow$ contradiction. Thus, limit dne.