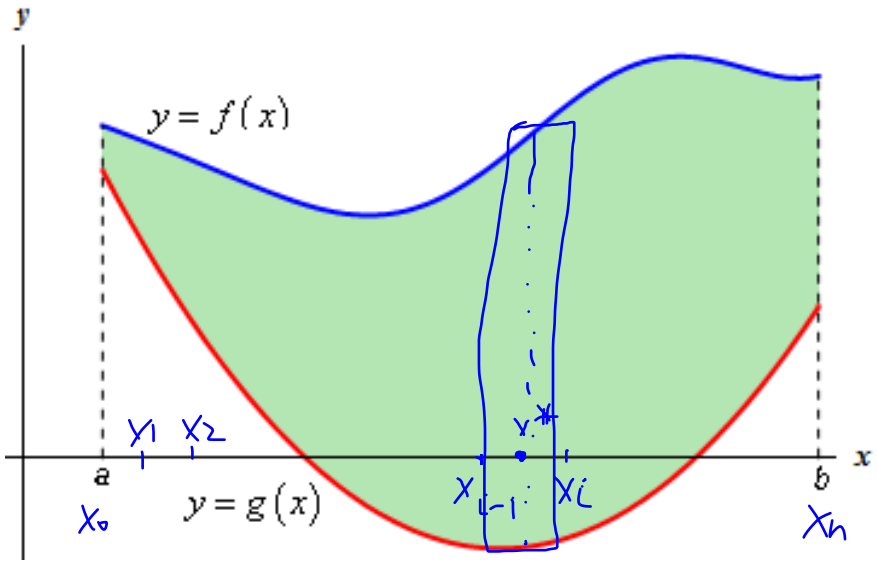
Lecture 11 (Application of Integration)

Areas between Curves

Let $f(x)$ and $g(x)$ be continuous and $f(x) \geq g(x)$ on $[a, b]$. Let's look at the region between $f(x)$ and $g(x)$ on $[a, b]$.



$$\Delta x = \frac{b-a}{n}$$

$$R_i = [f(x_i^*) - g(x_i^*)]\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

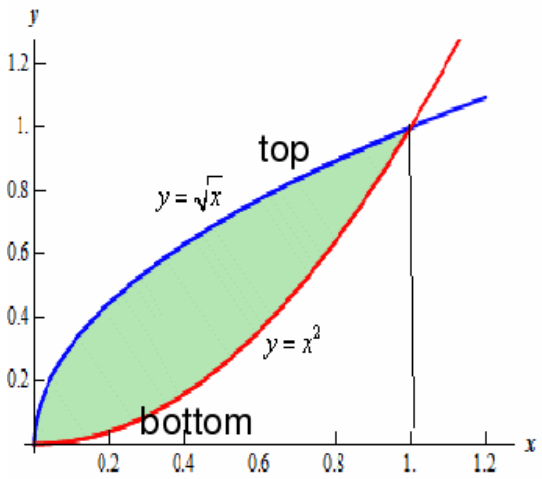
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

Definition: The **area** A of the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a, x = b$, where $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x)$ on $[a, b]$, is defined as

$$A = \int_a^b [f(x) - g(x)]dx = \int_a^b (y^T - y^B)\,dx$$

top, bottom

Example: Find the area of the region bounded by $y = \sqrt{x}, y = x^2$, and $x = 1$.

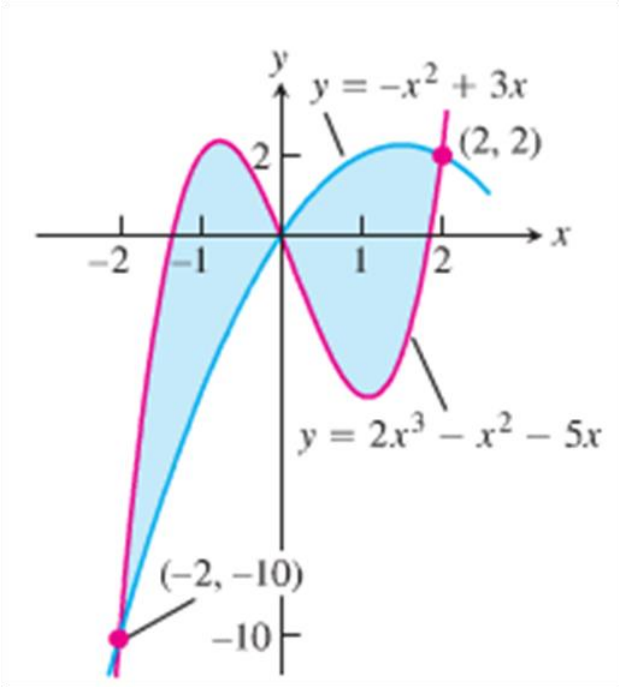


$$A = \int_0^1 [\sqrt{x} - x^2]\,dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Example: Find the area of the region enclosed by curves $y = 2x^3 - x^2 - 5x$ and $y = -x^2 + 3x$.



$$A = \int_{-2}^{0} \left[2x^3 - x^2 - 5x - (-x^2 + 3x)\right] dx + \int_{0}^{2} \left[-x^2 + 3x - (2x^3 - x^2 - 5x)\right] dx$$

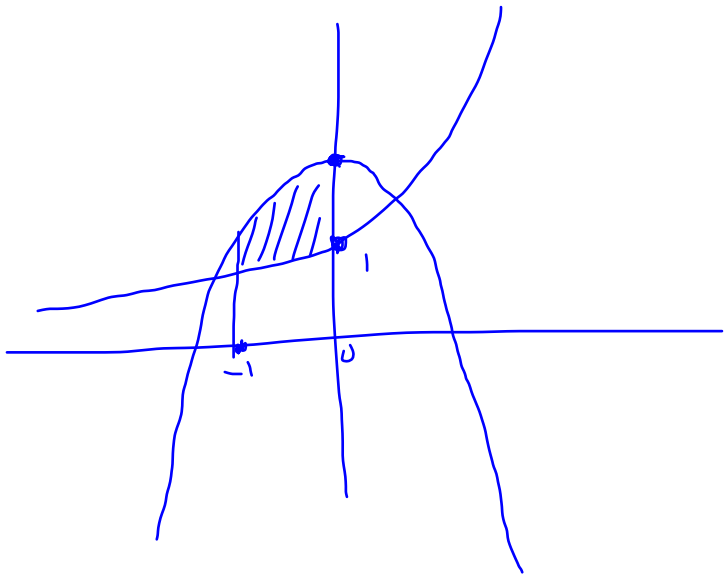
$$= \int_{-2}^{0} \left[2x^3 - 8x\right] dx + \int_{0}^{2} \left[-2x^3 + 8x\right] dx$$

$$= \left[\frac{x^4}{2} - 4x^2\right]_{-2}^{0} + \left[-\frac{x^4}{2} + 4x^2\right]_{0}^{2}$$

$$= -\frac{16}{2} + 16 - \frac{16}{2} + 16$$

$$= 32 - 16 = 16$$

Example: Find the area of the region enclosed by $y = e^x$, $y = 2 - x^2$, $x = -1$, and $x = 0$.



$e^{-1} = \frac{1}{e}$

$2 - (-1)^2 = 1$

$$A = \int_{-1}^{0} [2 - x^2 - e^x]\,dx$$

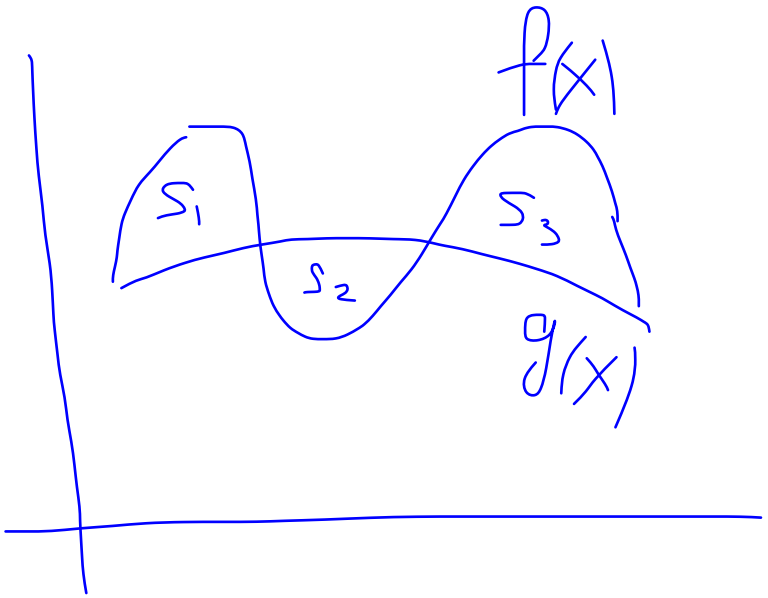
$$= \left[2x - \frac{x^3}{3} - e^x\right]_{-1}^{0} = -1 + 2 - \frac{1}{3} + e^{-1}$$

$$= \frac{2}{3} + \frac{1}{e}$$

In general, the area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is given by

$$A = \int_a^b |f(x) - g(x)|dx$$

Why?



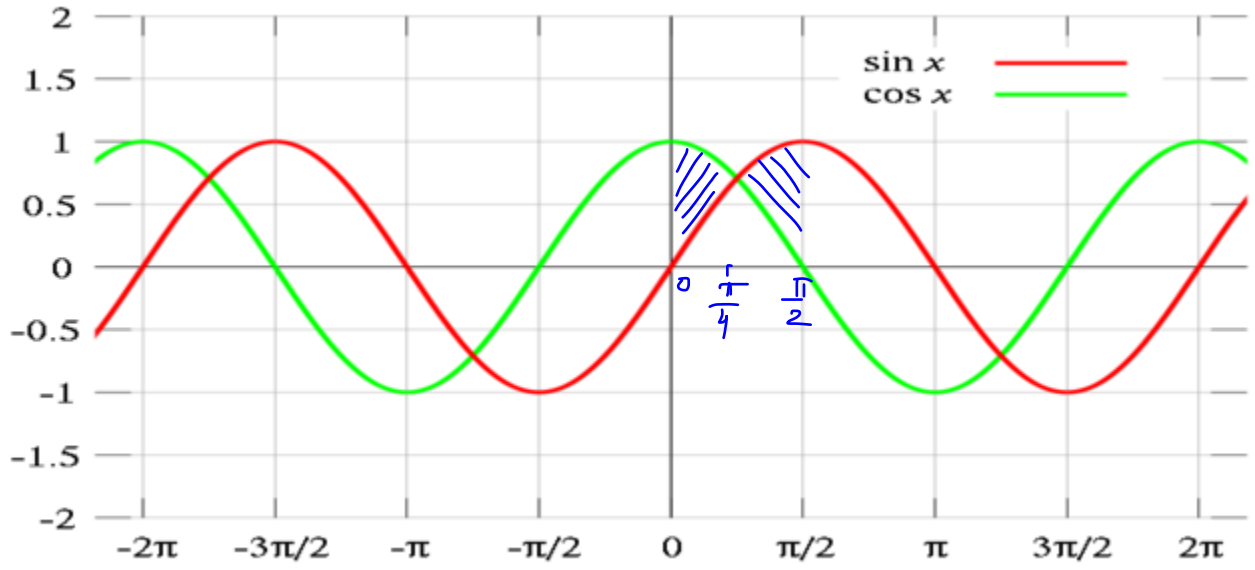
$S = S_1 + S_2 + S_3$

S_1, S_3 : $f(x) - g(x)$, $f(x) \geq g(x)$

S_2 : $g(x) - f(x)$, $g(x) \geq f(x)$

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x), & f(x) \geq g(x) \\ g(x) - f(x), & g(x) \geq f(x) \end{cases}$$

Example: Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.



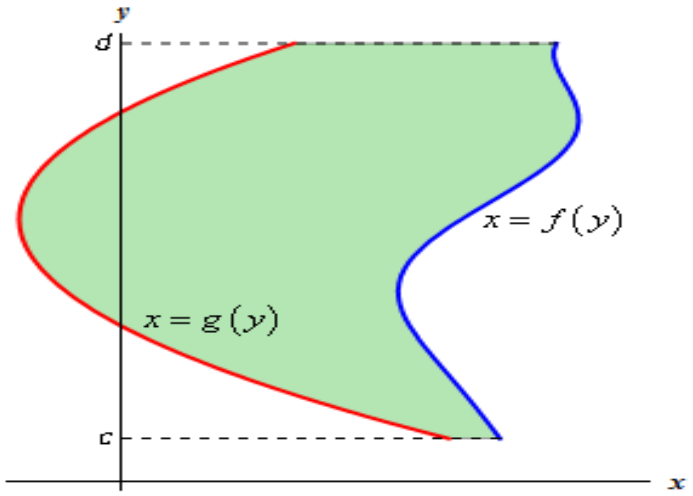
$$A = 2\int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= 2\left[\sin x + \cos x\right]_0^{\pi/4}$$

$$= 2\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1\right] = 2\sqrt{2} - 2$$

Very often the region is bounded by functions of y instead of x, i.e. $x = f(y)$ and $x = g(y)$. Then if $f(y) \geq g(y)$ on $[c, d]$, we define the area as

$$A = \int_c^d [f(y) - g(y)]dy$$

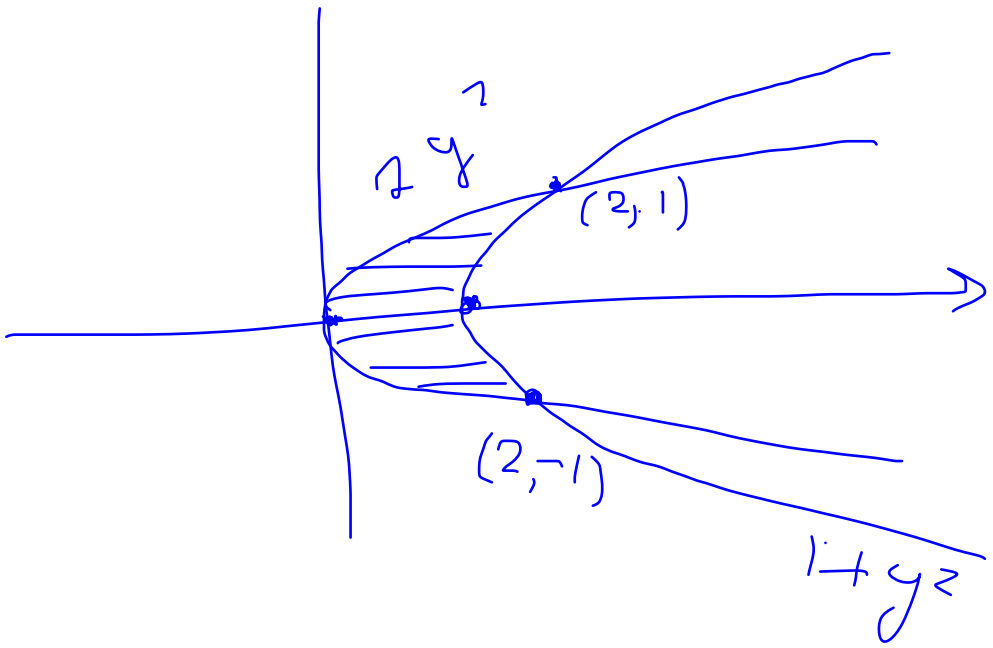


X_R - right boundary

X_L - left boundary

$$A = \int_c^d [X_R - X_L]\, dy$$

Example: Sketch the region enclosed by $x = 2y^2$ and $x = 1 + y^2$, and find its area.



$$2y^2 = 1 + y^2$$

$$y^2 = 1$$

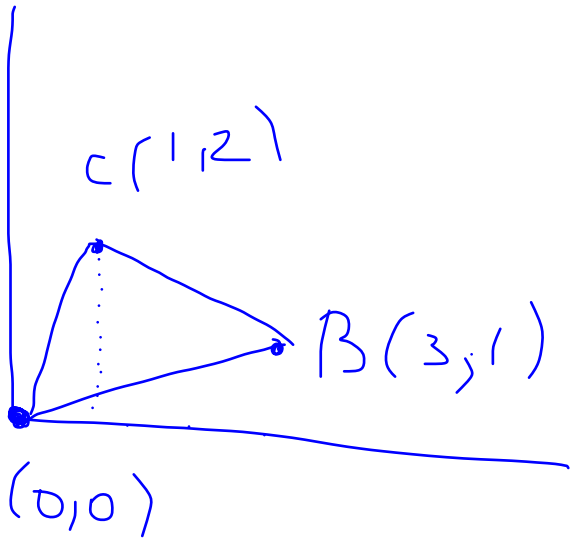
$$y = \pm 1$$

$$A = \int_{-1}^{1} \underbrace{[1 + y^2}_{x_R} - \underbrace{2y^2}_{x_L}] \, dy = \left[y - \frac{y^3}{3} \right]_{-1}^{1}$$

$$= 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$

Example: Find the area of the triangle with the given vertices:

(0, 0), (3, 1), (1, 2).



$AC: \; y = 2x$

$AB: \; y = \frac{1}{3}x$

$BC:$

$m_{BC} = \frac{2-1}{1-3} = -\frac{1}{2}$

$y - 2 = -\frac{1}{2}(x-1)$

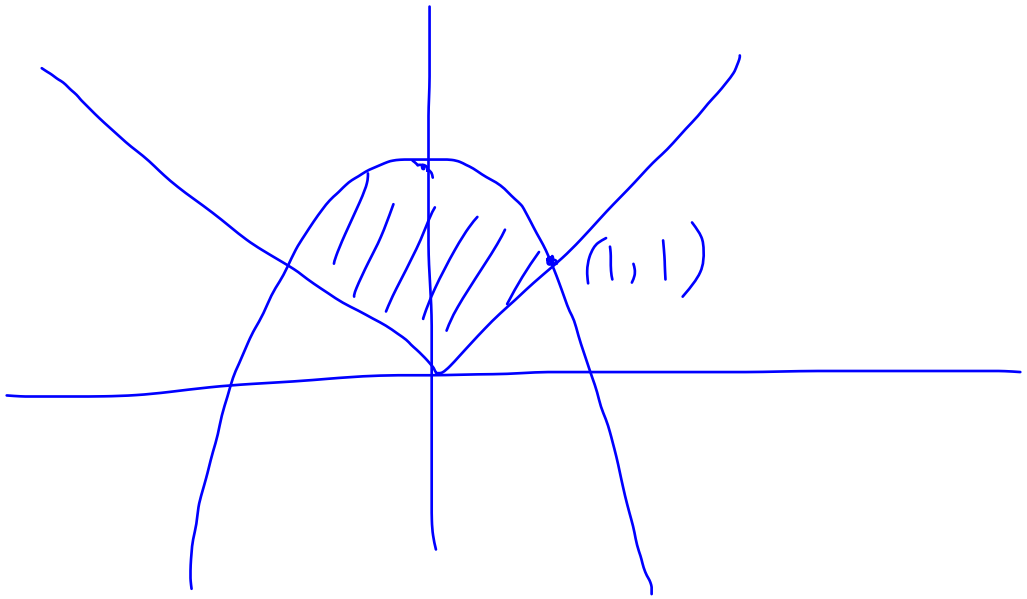
$BC: \; y = -\frac{1}{2}x + \frac{5}{2}$

$$A_{ABC} = \int_0^1 \left(2x - \frac{1}{3}x\right) dx + \int_1^3 \left(-\frac{1}{2}x + \frac{5}{2} - \frac{1}{3}x\right) dx$$

$$= \frac{5}{3}\frac{x^2}{2}\Big|_0^1 + \left[-\frac{5}{6}\cdot\frac{x^2}{2} + \frac{5}{2}x\right]_1^3$$

$$= 5/2$$

Example: Find the area enclosed by $y = |x|$ and $y = 2 - x^2$



$x > 0$, $x = 2 - x^2$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

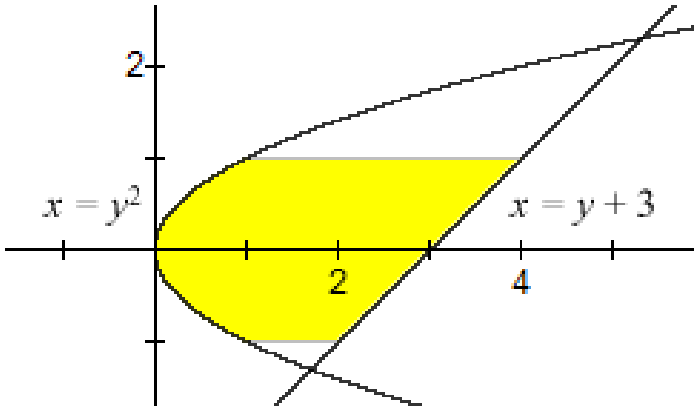
$x = 1$, $x = -2$

$$A = 2\int_0^1 [\underbrace{2 - x^2}_{y_T} - \underbrace{x}_{y_B}]\,dx$$

$$= 2\left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = 2\left[2 - \frac{1}{3} - \frac{1}{2}\right]$$

$$= 2 \cdot \frac{7}{6} = \frac{7}{3}$$

Example: Find the area enclosed by $x = y^2$ and $x = y + 3$ from $y = -1$ to $y = 1$.



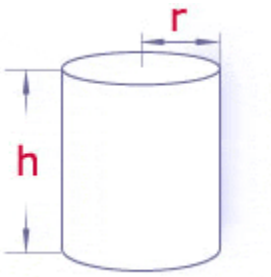
$$A = \int_{-1}^{1} \left[\underbrace{y+3}_{x_R} - \underbrace{y^2}_{x_L} \right] dy$$

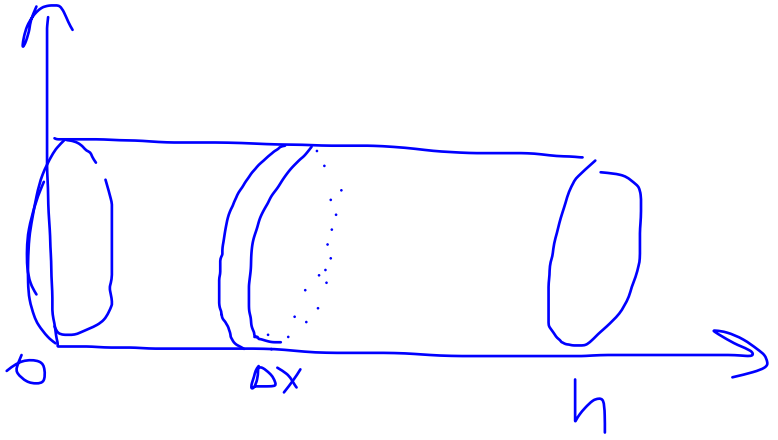
$$= \left[\frac{y^2}{2} + 3y - \frac{y^3}{3} \right]_{-1}^{1} =$$

$$= \frac{1}{2} + 3 - \frac{1}{3} - \frac{1}{2} + 3 - \frac{1}{3} = 6 - \frac{2}{3} = \frac{16}{3}$$

Volumes

To find volumes, we shall use similar procedure as with areas. We shall start with a simple example: a **cylinder**.





It is known that $V = \pi r^2 h$.

To prove it, we cut the cylinder into thin disks (cross-sections).

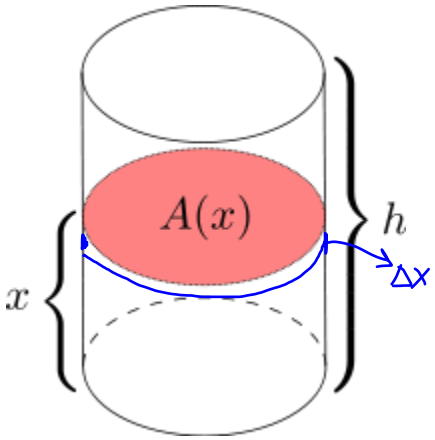
The cross-sectional area is denoted by $A(x) = \pi r^2$. Thus, the volume of each disk is given by $A(x)\Delta x$.

So $V \approx \sum A(x_i)\Delta x$.

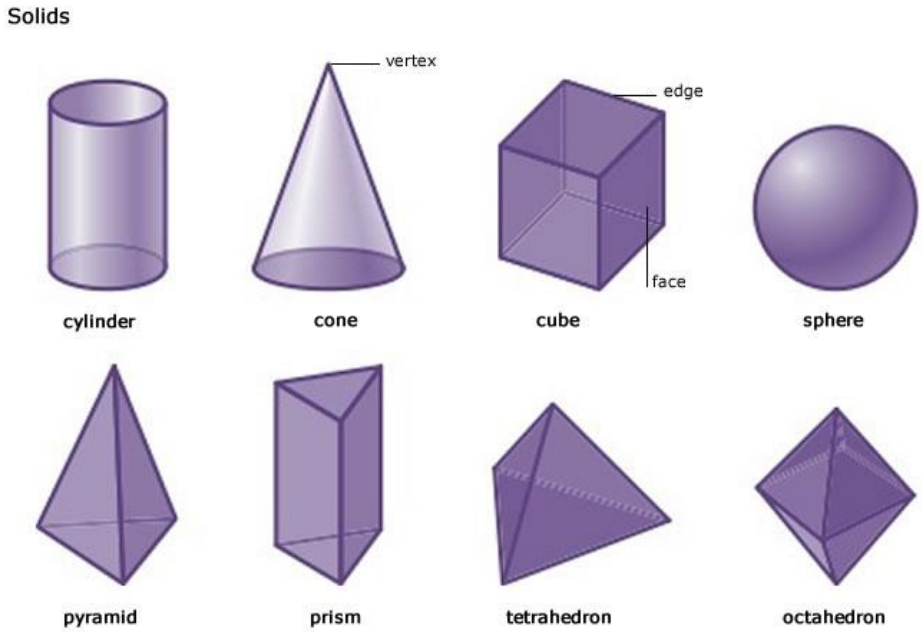
$V = \lim_{\Delta x \to 0} \sum A(x_i)\Delta x$

Then $V = \int_0^h A(x)dx = \int_0^h \pi r^2 dx = \pi r^2 \int_0^h dx$

$= \pi r^2 h$



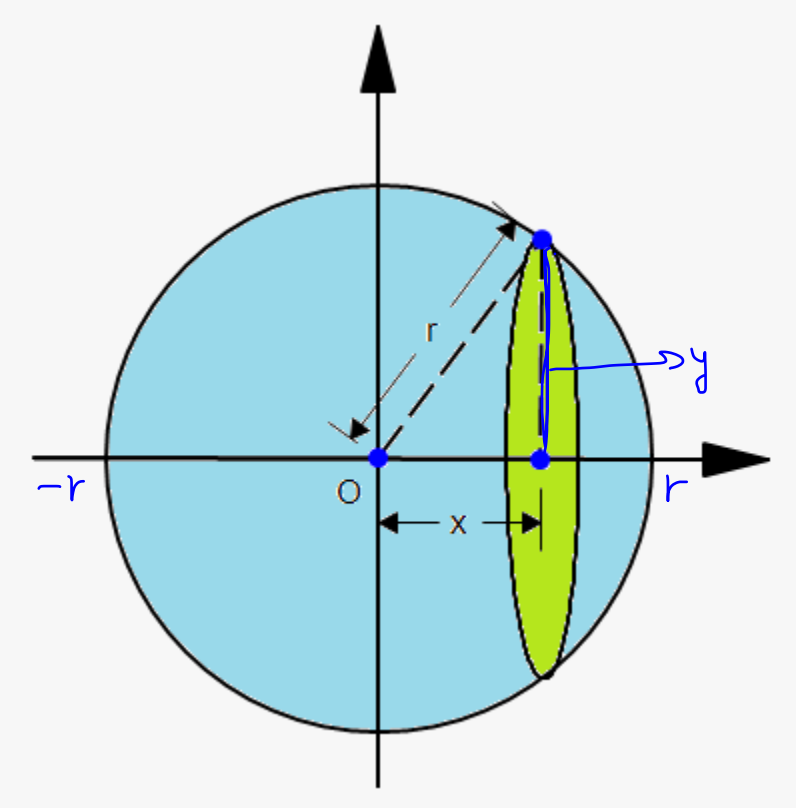
We can use similar principle to find volumes of many other solids:



Definition: Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S (perpendicular to the x-axis) is $A(x)$, then the **volume** of S is

$$V = \int_a^b A(x)dx$$

Example: Find the volume of a sphere of radius r.



$$x^2+y^2=r^2$$

$$y=\sqrt{r^2-x^2}$$

$$A(x)=\pi y^2=\pi(r^2-x^2)$$

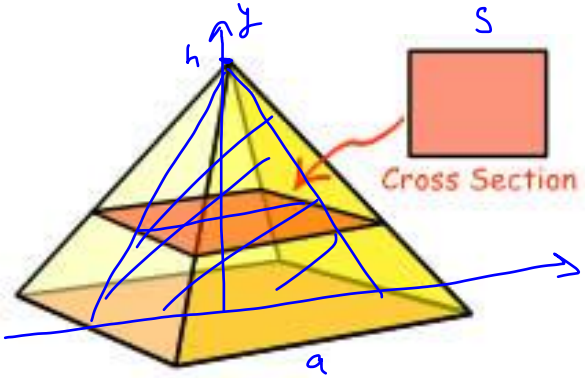
$$V=\int_{-r}^{r} A(x)\,dx$$

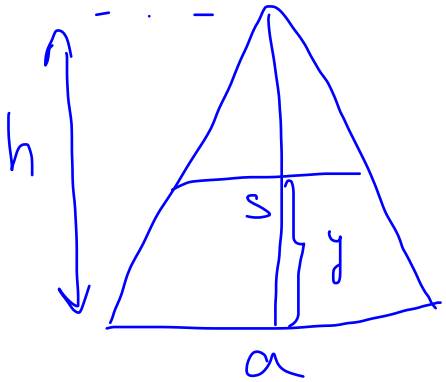
$$=\pi\int_{-r}^{r}(r^2-x^2)\,dx$$

$$=2\pi\int_{0}^{r}(r^2-x^2)\,dx=2\pi\left[r^2x-\frac{x^3}{3}\right]_0^r$$

$$=2\pi\left[r^3-\frac{r^3}{3}\right]=2\pi\cdot\frac{2r^3}{3}=\frac{4}{3}\pi r^3$$

Example: Find the volume of a pyramid with a squared base of side a and height h.





$A(y) = s^2$ (Similar Δs) $\frac{a}{s} = \frac{h}{h-y}$

$= \frac{a^2}{h^2}(h-y)^2$ $s = \frac{a(h-y)}{h}$

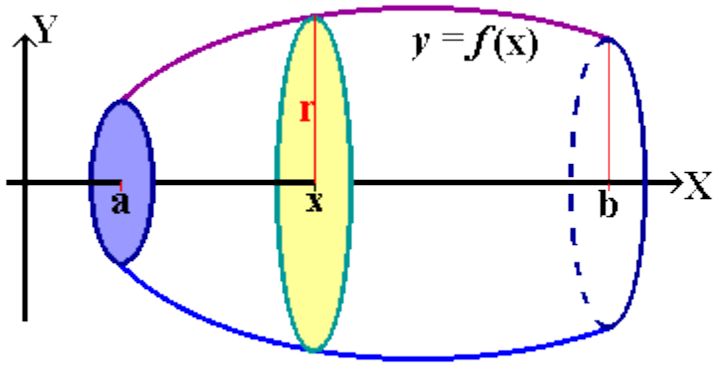
$$V = \int_0^h A(y)\,dy = \frac{a^2}{h^2}\int_0^h (h-y)^2\,dy = -\frac{a^2}{h^2}\frac{(h-y)^3}{3}\Big|_0^h$$

$$= \frac{a^2}{h^2}\frac{h^3}{3} = \frac{a^2 h}{3}$$

Volumes of Solids of Revolutions

Volumes of revolution are obtained by revolving a two-dimensional region about an axis of rotation.

Consider the graph of a function $y = f(x)$ defined on $[a, b]$. Suppose that the region between $f(x)$ and the x-axis is revolved about the x-axis[1]:



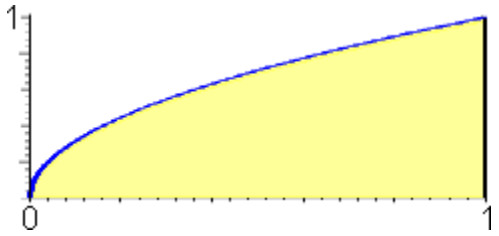
$r = f(x)$

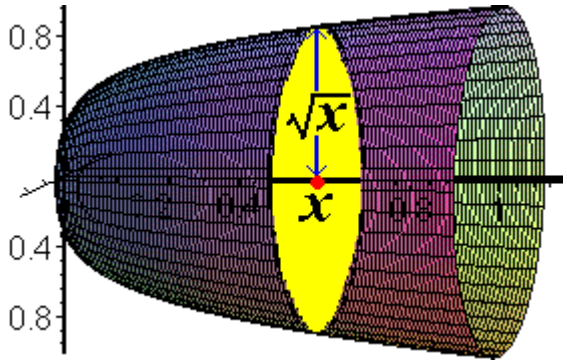
$$A(x) = \pi r^2 = \pi [f(x)]^2$$

$$V = \int_a^b A(x)\,dx = \pi \int_a^b [f(x)]^2\,dx$$

[1] http://www2.bc.cc.ca.us/resperic/Math6A/Lectures/ch6/2/disk.htm

Example: Find the volume of the solid of revolution formed by rotating the region bounded by the x-axis and the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 1$, about the x-axis.

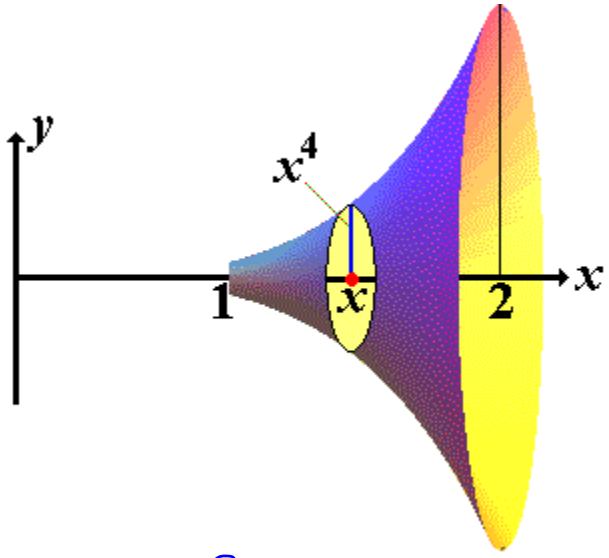




$$V = \pi \int_0^1 (\sqrt{x})^2 \, dx = \pi \int_0^1 x \, dx$$

$$= \pi \frac{x^2}{2} \Big|_0^1 = \frac{\pi}{2}$$

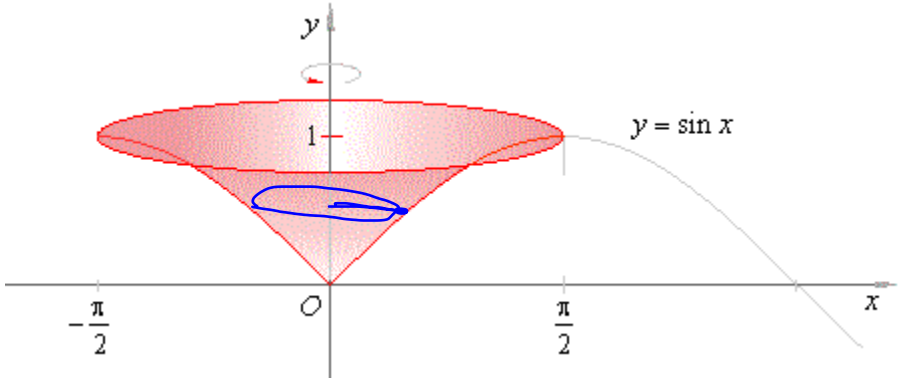
Example: Find the volume of the solid of revolution formed by rotating the region bounded by the x-axis and the graph of $= x^4$, from $x = 1$ to $x = 2$, about the x-axis.



$$V = \pi \int_1^2 (x^4)^2 \, dx = \pi \left. \frac{x^9}{9} \right|_1^2$$

$$= \frac{\pi}{9} \left[2^9 - 1 \right] = \frac{511\pi}{9}$$

Example[2]: Find the volume of a solid of revolution generated by $y = \sin x$ between $x = 0$ and $x = \pi/2$ revolving around the y-axis.



$x = \sin^{-1} y$

$0 \le y \le 1$

$$V = \pi \int_0^1 [\sin^{-1} y]^2 \, dy = \pi \left[y (\sin^{-1} y)^2 \right]_0^1 - 2 \int_0^1 \frac{y \sin^{-1} y}{\sqrt{1-y^2}} \, dy$$

$u = [\sin^{-1} y]^2$ $\quad dv = dy$

$du = \dfrac{2 \sin^{-1} y}{\sqrt{1-y^2}} \, dy$ $\quad v = y$

[2] http://www.nabla.hr/CL-DefiniteIntAppl4.htm

$$= \pi \left[\left(\frac{\pi}{2}\right)^2 - \int_0^1 \frac{2y \sin^{-1} y}{\sqrt{1-y^2}}\, dy \right]$$

$u = \sin^{-1} y$

$du = \frac{1}{\sqrt{1-y^2}}\, dy$

$dv = \frac{y\, dy}{\sqrt{1-y^2}}$

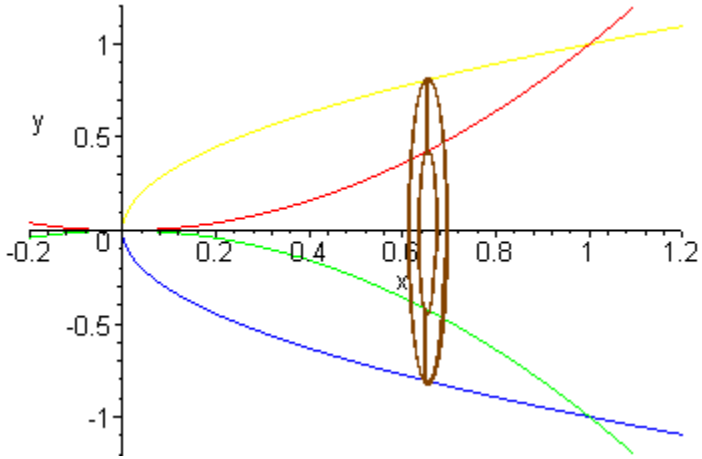
$v = -\sqrt{1-y^2}$

$$= \pi \left[\left(\frac{\pi}{2}\right)^2 + 2 \sin^{-1} y \cdot \sqrt{1-y^2} \Big|_0^1 - 2 \int_0^1 \frac{\sqrt{1-y^2}}{\sqrt{1-y^2}}\, dy \right]$$

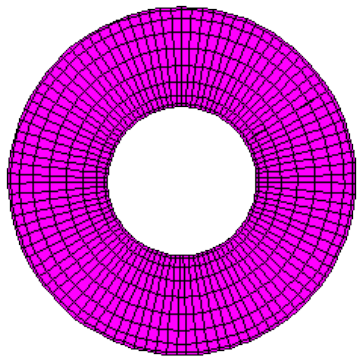
$$= \pi \left[\left(\frac{\pi}{2}\right)^2 + 0 - 2y \Big|_0^1 \right]$$

$$= \pi \left[\frac{\pi^2}{4} - 2 \right]$$

Example: Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. $x^2 = \sqrt{x}$ $x = 0, 1$

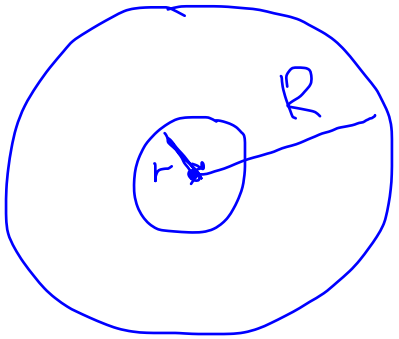


The cross-section of this type is called a **washer**.



The area of the washer is equal to the area of the outer disk minus the area of the inner disk:

$$A = \pi(R^2 - r^2)$$



R = top function $= \sqrt{x}$

r = bottom function $= x^2$

$$A(x) = \pi\left[(\sqrt{x})^2 - (x^2)^2\right]$$

$$V = \int_a^b A(x)\,dx = \pi\int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right]dx$$

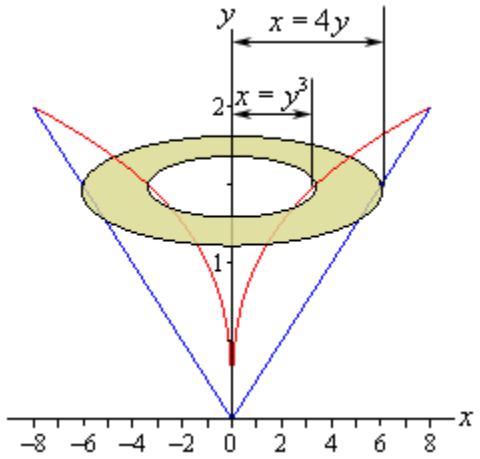
$$= \pi\int_0^1 \left(x - x^4\right)dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}$$

Volume by washers: If the region below $y = f(x)$, above $y = g(x)$, and between the lines $x = a$ and $x = b$ is revolved around the x-axis, then the volume of the resulting solid is given by

$$V = \pi\int_a^b [f(x)^2 - g(x)^2]dx$$

Example[3]: Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the y-axis.



$x = y^3$

$x = 4y$

$y^3 = 4y$

$y^3 - 4y = 0$

$y(y^2 - 4) = 0$

$y = 0, \pm 2$

$y \geq 0$

$A(y) = \pi[(\text{right function})^2 - (\text{left function})^2]$

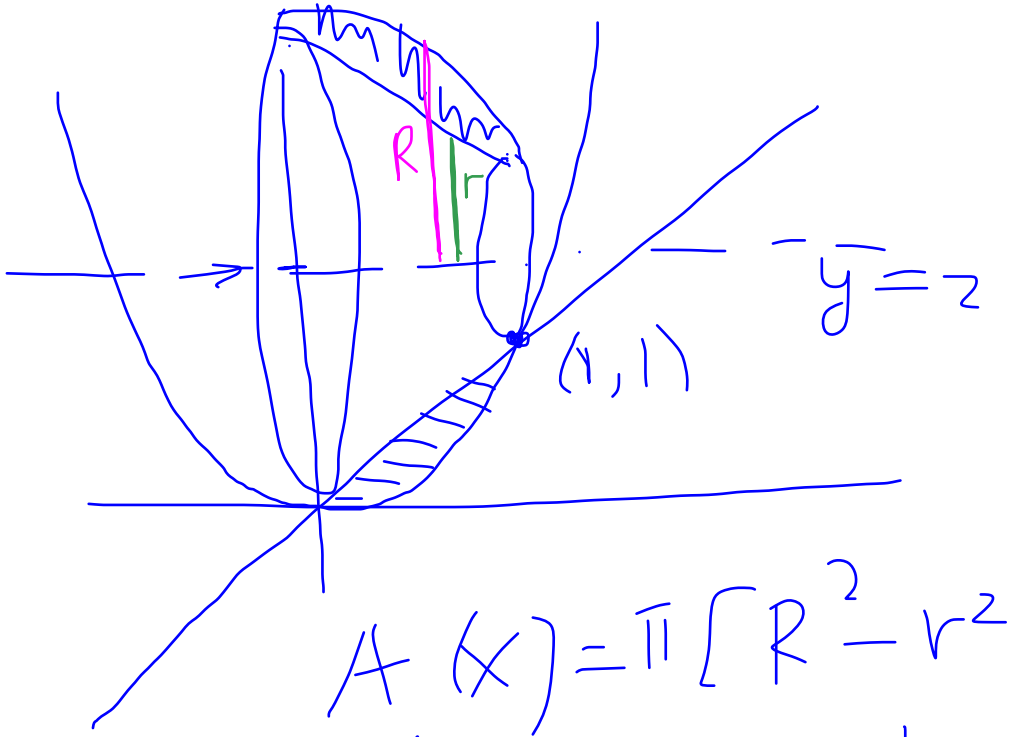
$= \pi[(4y)^2 - (y^3)^2]$

$V = \int_a^b A(y)\,dy = \pi \int_0^2 [16y^2 - y^6]\,dy = \pi\left[\frac{16}{3}y^3 - \frac{y^7}{7}\right]_0^2$

$= \frac{512\pi}{21}$

[3] http://tutorial.math.lamar.edu/Classes/CalcI/VolumeWithRings.aspx

Example: Find the volume of the solid obtained by rotating the region enclosed by $y = x$ and $y = x^2$ about the line $y = 2$.



$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

$$A(x) = \pi\left[R^2 - r^2\right] = \pi\left[(x^2-2)^2 - (x-2)^2\right]$$

$$V = \int_0^1 A(x)\,dx = \pi\int_0^1 \left[x^4 - 4x^2 + y - x^2 + 4x - 4\right]dx$$

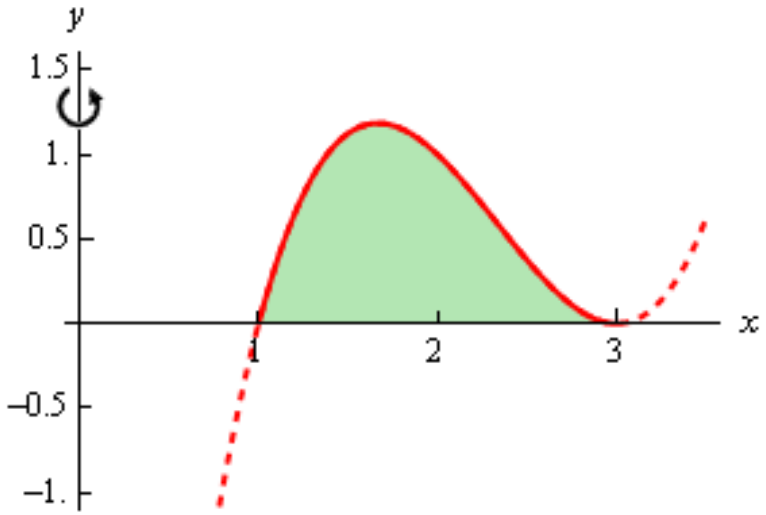
$$= \pi\left[\frac{x^5}{5} - 5\,\frac{x^3}{3} + 4\cdot\frac{x^2}{2}\right]_0^1 = \frac{8\pi}{15}$$

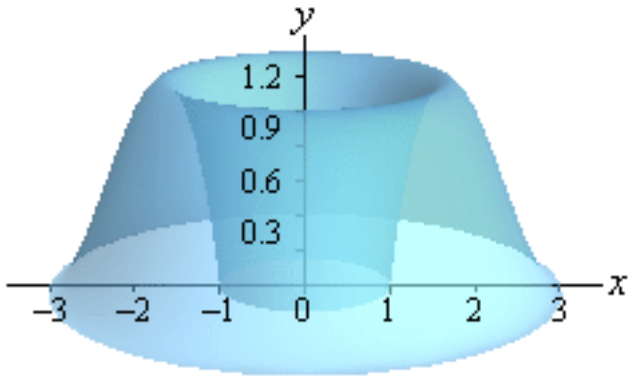
6.3

Volumes by Cylindrical Shells → Skip

Unfortunately, we can't always use discs or washers.

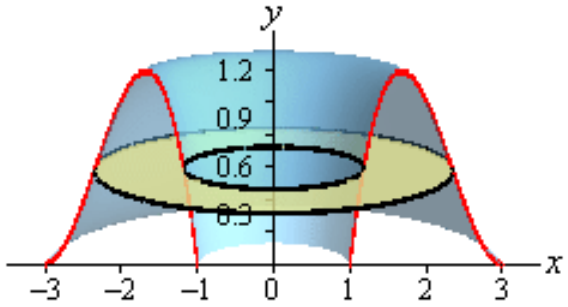
Example[4]: Determine the volume of the solid obtained by rotating the region bounded by $y = (x-1)(x-3)^2$ and the x-axis about the y-axis.





Let's try to use washers.

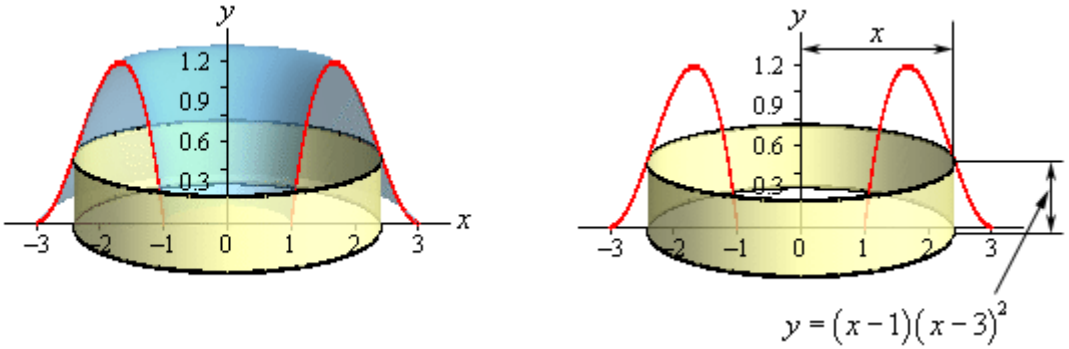
[4] http://tutorial.math.lamar.edu/Classes/CalcI/VolumeWithCylinder.aspx



Several problems arise:

- Both the inner and outer radius are defined by the same function.
- In order to use washers we would need to put this function into the form $x = f(y)$. That is NOT easy in general for a cubic polynomial and in other cases may not even be possible to do.
- The last problem with rings in this case is not so much a problem as it's just added work. If we were to use rings the limit would be y limits and this means that we will need to know how high the graph goes. To this point the limits of integration have always been intersection points that were fairly easy to find. However, in this case the highest point is not an intersection point, but instead a maximum. Finding it will take some time.

So we need to find a different way to cut the solid that will give us a cross-sectional area that we can work with. One way of doing this is to think of our solid as a lump of cookie dough and instead of cutting it perpendicular to the axis of rotation we center a cylindrical cookie cutter on the axis of rotation and push it down into the solid. Like this:



This will give us a **cylindrical shell** and we can easily find its surface area:

$$A(x) = 2\pi rh$$

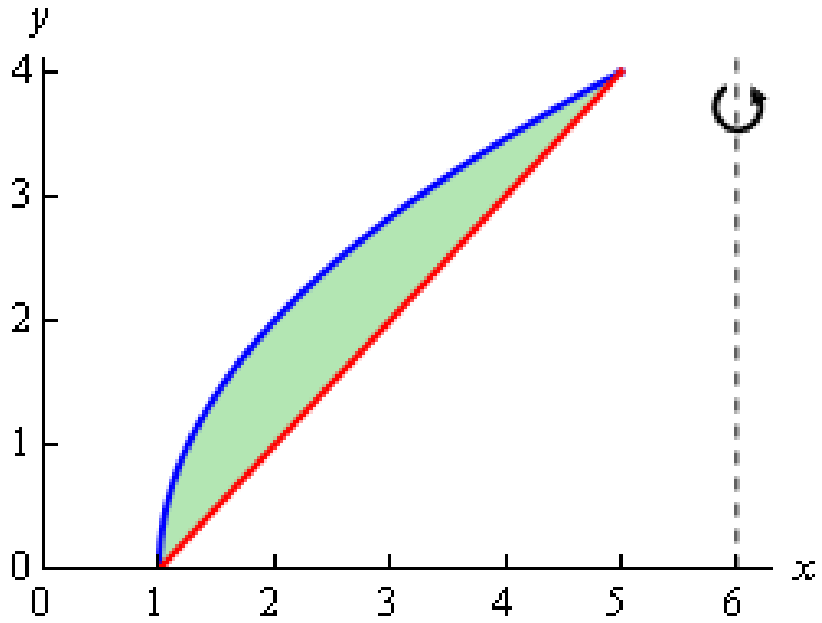
The first cylinder will cut into the solid at $x = 1$ and as we increase x to $x = 3$ we will completely cover both sides of the solid since expanding the cylinder in one direction will automatically expand it in the other direction as well.

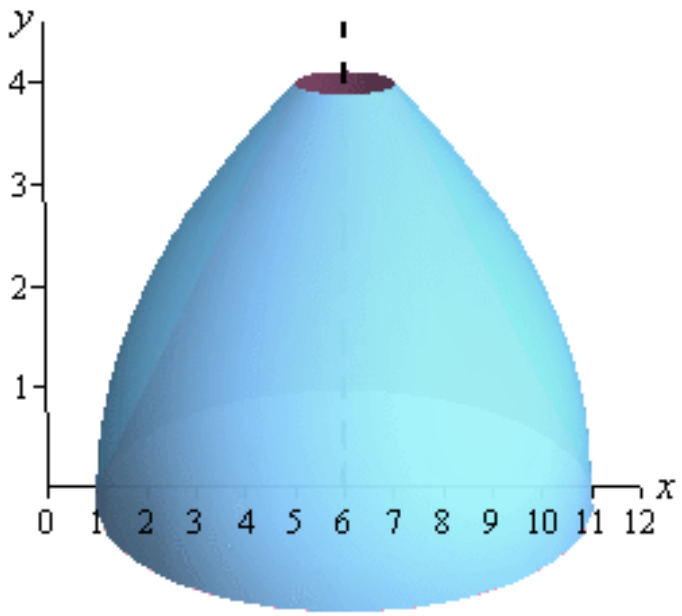
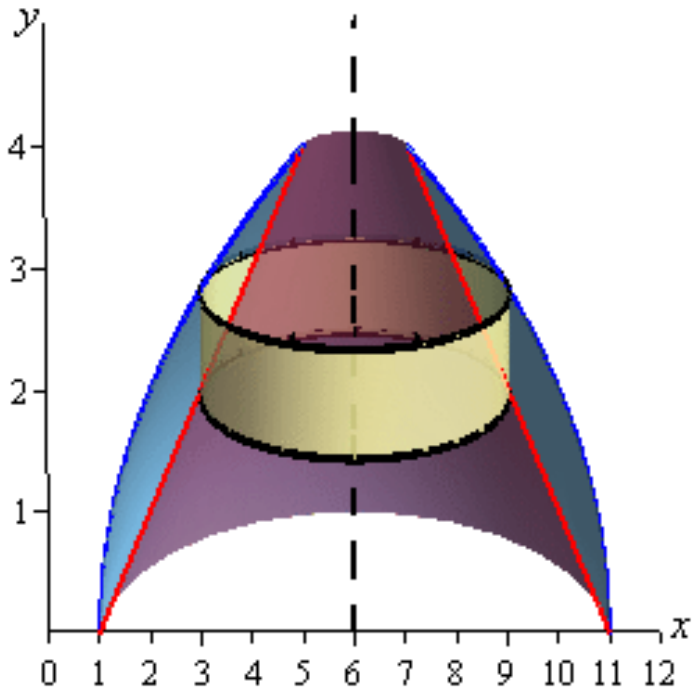
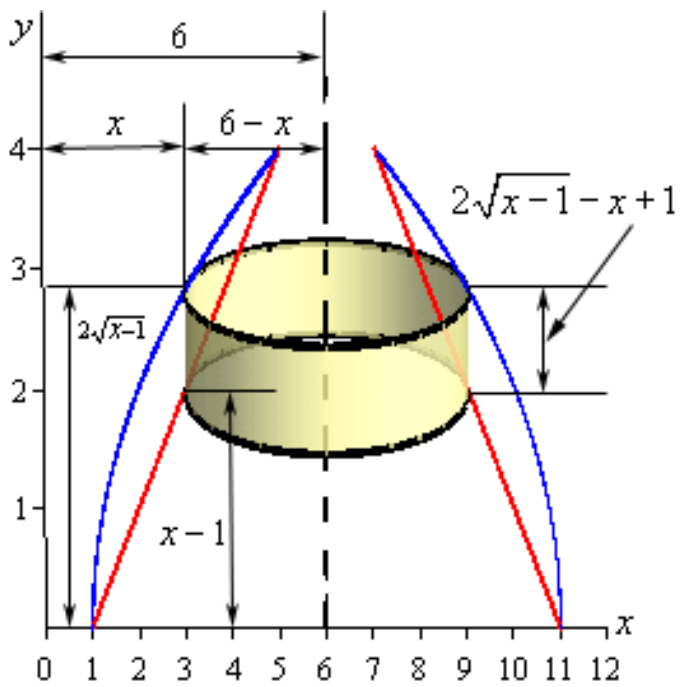
The volume of this solid is then,

$$V = \int_a^b A(x)dx$$

The method used here is called the **method of cylindrical shells**.

Example: Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = 6$.





In this example both methods can be used (shells or washers)