Lecture 10 (Techniques of Integration continued)

Integration of Rational Functions by Partial Fractions

Recall: A rational function is the quotient of two polynomials, i.e.

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x), Q(x)$ are polynomials.

A rational function is called **proper** if the degree of P is less than the degree of Q, i.e. $\deg(P) < \deg(Q)$.

If $f(x)$ is **improper**, i.e. $\deg(P) \geq \deg(Q)$, then we can use long division and write $f(x)$ as the sum of a polynomial and a proper rational funtion:

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x), R(x)$ are polynomials and $\deg(R) < \deg(Q)$.

Example: Find $\int \frac{x^2-2x+2}{x-1}\,dx$

$$
\begin{array}{r|l}
 & x - 1 \\
\hline
x-1 & x^2 - 2x + 2 \\
- & x^2 - x \\
\hline
 & -x + 2 \\
- & -x + 1 \\
\hline
 & 1
\end{array}
$$

$$\frac{x^2-2x+2}{x-1} = (x-1) + \frac{1}{x-1}$$

$$= \int (x-1)\,dx + \int \frac{1}{x-1}\,dx$$

$$= \frac{x^2}{2} - x + \ln|x-1| + C$$

Sometimes more than one step is required.

In general, if we want to integrate a rational function, we need to

- Write $f(x) = \frac{P(x)}{Q(x)}$ as the sum of a polynomial and a proper rational function: $S(x) + \frac{R(x)}{Q(x)}$
- Factor $Q(x)$ as far as possible.
- Express the proper rational function $\frac{R(x)}{Q(x)}$ as a sum of **partial fractions** of the form:

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

There are several cases that might occur. Let's consider each separately.

Case I: $Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$ (no factor is repeated)

Then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}$$

Demonstration:

$$\frac{x}{x^2 - 9} = \frac{x}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} = \frac{1}{2(x-3)} + \frac{1}{2(x+3)}$$

$$\frac{x}{(x-3)(x+3)} = \frac{A(x+3) + B(x-3)}{(x-3)(x+3)}$$

$$x = A(x+3) + B(x-3)$$

$$x = (A+B)x + (3A - 3B)$$

$$A + B = 1 \Rightarrow A = B = \frac{1}{2}$$

$$3A - 3B = 0 \Rightarrow A = B$$

Example: Calculate $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$ $= \int \left[x+1 + \frac{3x-4}{x^2-x-6} \right] dx$

$(x-3)(x+2)$

$x+1$

$x^2-x-6 \,\big)\, x^3 - 4x - 10$

$-\quad x^3 - x^2 - 6x$

$x^2 + 2x - 10$

$-\quad x^2 - x - 6$

$3x - 4$

$$\frac{3x-4}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$3x-4 = A(x+2) + B(x-3)$

$3x-4 = (A+B)x + (2A-3B)$

$A+B=3 \Rightarrow A=3-B \Rightarrow A=3-2=1$

$2A-3B=-4 \Rightarrow 2(3-B)-3B=-4 \Rightarrow B=2$

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 \left[x+1+\frac{1}{x-3}+\frac{2}{x+2} \right] dx$$

$$= \left[\frac{x^2}{2} + x + \ln|x-3| + 2\ln|x+2| \right]_0^1$$

$$= \frac{1}{2} + 1 + \ln 2 + 2\ln 3 - \ln 3 - 2\ln 2$$

$$= \frac{3}{2} + \ln 3 - \ln 2$$

$$= \frac{3}{2} + \ln\frac{3}{2}$$

Case II: some factors of $Q(x)$ are repeated, e.g. $Q(x) = (ax+b)^k$. Then we would write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

Demonstration:

$$\frac{x^2+1}{x^3(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$$

$$x^2+1 = A\,x^2(x+1) + Bx(x+1) + C(x+1) + Dx^3$$

x^3 : $0 = A + D \Rightarrow D = -2$

x^2 : $1 = A + B \Rightarrow A = 2$

x : $0 = B + C \Rightarrow B = -1$

const : $1 = C$

$$\frac{x^2+1}{x^3(x+1)} = \frac{2}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{2}{x+1}$$

Example: Calculate $\int \frac{x^2-5x+16}{(2x+1)(x-2)^2} = \int \left[\frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\right] dx$

$$x^2-5x+16 = A(x-2)^2 + B(x-2)(2x+1) + C(2x+1)$$

$$A(x^2-4x+4) + B(2x^2-3x-2) + C(2x+1)$$

$$(A+2B)x^2 + (-4A-3B+2C)x + (4A-2B+C)$$

x^2: $1 = A+2B$ $\Longrightarrow$ $1 = A - 74 + 24A$

$25A = 75$

$\boxed{A=3}$

x: $-5 = -4A-3B+2C$

$+$

const: $16 = 4A-2B+C$ $\cdot(-2)$

$-37 = -12A + B$

$B = -37 + 12A$

$B = -37 + 12 \cdot 3$

$\boxed{B = -1}$

$C = 16 - 4A + 2B$

$= 16 - 4 \cdot 3 + 2(-1)$

$\boxed{C = 2}$

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2} = \int \left[\frac{3}{2x+1} - \frac{1}{x-2} + \frac{2}{(x-2)^2} \right] dx$$

$$= \frac{3}{2} \ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C$$

$$\frac{1}{x^2} \longrightarrow -\frac{1}{x}$$

Case III: $Q(x)$ has irreducible quadratic factors (not repeated), i.e. factors of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$. Then we add one more term to the partial fractions of $\frac{R(x)}{Q(x)}$:

$$\frac{Ax + B}{ax^2 + bx + c}$$

Demonstration:

$$\frac{2x}{(x+1)(x^2+3)(x^2+1)} =$$

$$= \frac{A}{x+1} + \frac{Bx + C}{x^2+3} + \frac{Dx + E}{x^2+1}$$

Useful formula:

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c}$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

Proof:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1} = \frac{1}{a}\int \frac{\frac{1}{a}\,dx}{\left(\frac{x}{a}\right)^2+1}$$

$$u = \frac{x}{a}$$

$$du = \frac{1}{a}\,dx$$

$$= \frac{1}{a}\int \frac{du}{u^2+1} = \frac{1}{a}\tan^{-1} u + C$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Example: Evaluate $\int \frac{2x^2-x+4}{x^3+4x}dx$ $= \int \frac{2x^2-x+4}{x(x^2+4)}dx$

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2-x+4 = A(x^2+4) + (Bx+C)x$$

x^2: $2 = A+B \Rightarrow B = 2-A = 1$

x: $-1 = C$

const: $4 = 4A \Rightarrow A = 1$

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \left[\frac{1}{x} + \frac{x-1}{x^2+4}\right] dx$$

$$= \int \left[\frac{1}{x} + \underbrace{\frac{x}{x^2+4}} - \underbrace{\frac{1}{x^2+4}}_{\frac{1}{x^2+2^2}}\right] dx$$

$$u = x^2+4$$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$= \int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{u}\,du - \int \frac{1}{x^2+2^2}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

Case IV: some factors of $Q(x)$ are repeated, e.g. $(ax^2+bx+c)^k$, where $b^2-4ac<0$. Then we would write

$$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

Demonstration:

$$\frac{x^3+2}{x(x^2+x+1)(2x^2+1)^2}=$$

$$=\frac{A}{x}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{2x^2+1}+\frac{Fx+G}{(2x^2+1)^2}$$

<u>Example</u>: Evaluate $\int \frac{1}{x(x^2+4)^2}\,dx$

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$$

$$A(x^4+8x^2+16) + (Bx+C)(x^3+4x) + (Dx+E)x$$

x^4: $0 = A + B \Rightarrow B = -A = -\frac{1}{16}$

x^3: $0 = C$

x^2: $0 = 8A + 4B + D \Rightarrow D = -8A - 4B = -8\cdot\frac{1}{16} - 4\cdot\left(-\frac{1}{16}\right)$

$= -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

x: $0 = 4C + E \Rightarrow E = 0$

const: $1 = 16A \Rightarrow A = \frac{1}{16}$

$$\int \frac{1}{x(x^2+4)^2}\,dx = \int \left[\frac{1}{16x} - \frac{x}{16(x^2+4)} - \frac{x}{4(x^2+4)^2} \right] dx$$

$$u = x^2+4$$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$= \frac{1}{16}\int \frac{1}{x}\,dx - \frac{1}{32}\int \frac{1}{u}\,du - \frac{1}{8}\int \frac{1}{u^2}\,du$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln|u| + \frac{1}{8}\frac{1}{u} + C$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)^2} + C$$

Example: $\int \frac{x^4+3x^2+1}{x^5+5x^3+5x} \, dx$

$x(x^4+5x^2+5)$

$(x^5+5x^3+5x)'$

$= 5x^4 + 15x^2 + 5$

$u = x^5+5x^3+5x$

$du = (5x^4+15x^2+5)dx$

$= 5(x^4+3x^2+1)dx$

$\frac{1}{5}du = (x^4+3x^2+1)dx$

$= \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\ln|u| + C$

$= \frac{1}{5}\ln|x^5+5x^3+5x| + C$

Example (Rationalizing substitution): Find $\int \frac{dx}{2\sqrt{x+3}+x}$

$$u = \sqrt{x+3}$$

$$u^2 = x+3 \Rightarrow x = u^2 - 3$$

$$2u\,du = dx$$

$$= \int \frac{2u\,du}{2u + u^2 - 3} = \int \frac{2u\,du}{u^2+2u-3}$$

$$(u+3)(u-1)$$

$$\frac{2u}{(u+3)(u-1)} = \frac{A}{u+3} + \frac{B}{u-1}$$

$$2 = A + B$$

$$0 = -A + 3B$$

$$2u = A(u-1) + B(u+3)$$

$$B = \frac{1}{2},\ A = 2 - \frac{1}{2} = \frac{3}{2}$$

$$= \int \frac{3}{2(u+3)}\,du + \int \frac{1}{2(u-1)}\,du$$

$$= \frac{3}{2} \ln|u+3| + \frac{1}{2} \ln|u-1| + c$$

$$= \frac{3}{2} \ln\left|\sqrt{x+3} + 3\right| + \frac{1}{2} \ln\left|\sqrt{x+3} - 1\right| + c$$

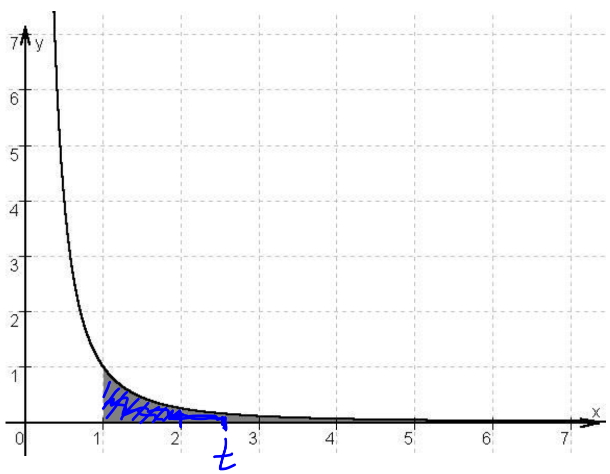
(For **Strategy for Integration** please read section 7.5 from the textbook)

Improper Integrals

Recall: a definite integral $\int_a^b f(x)dx$. Here we have a function f definified and continuous on a finite interval $[a, b]$. In this lecture we shall consider a case when the interval is infinite and function might not be continuous on an interval.

In these cases the integral is called an **improper integral**.

Example: $y = \frac{1}{x^2}, x > 0$



$$\int_1^\infty \frac{1}{x^2}dx$$

$$\int_1^2 \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^2 = -\frac{1}{2}+1 = \frac{1}{2}$$

$$\int_1^t \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^t = -\frac{1}{t}+1$$

$$A = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}\left[-\frac{1}{t}+1\right]$$

$$= 1$$

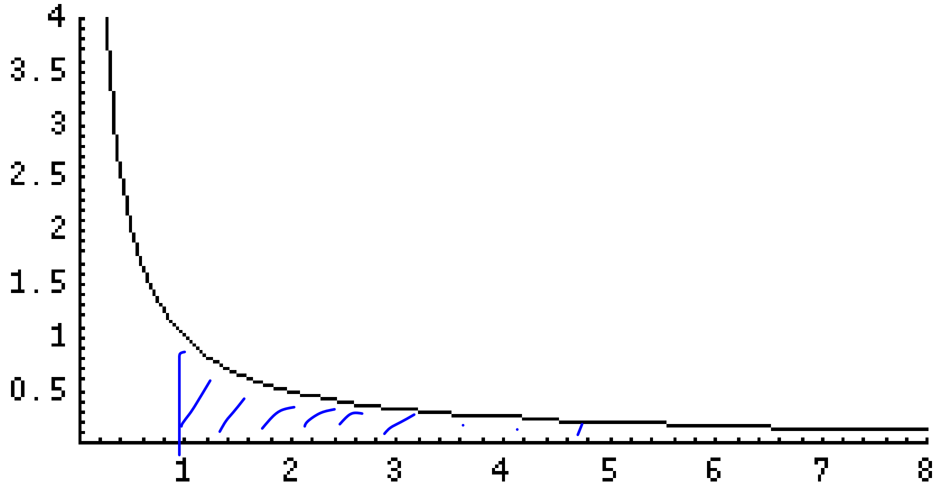
Improper Integrals Over Infinite Intervals

Definition:

- Let f be a continuous function defined on $[a, \infty)$. Then for $t \geq a$ define
$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$
provided that this limit exists.
- Let f be a continuous function defined on $(-\infty, b]$. Then for $t \leq b$ define
$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$
provided that this limit exists.

  The improper integrals $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{b} f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.
- If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ are convergent, then we can write
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

Example: $y = \frac{1}{x},\ x > 0$



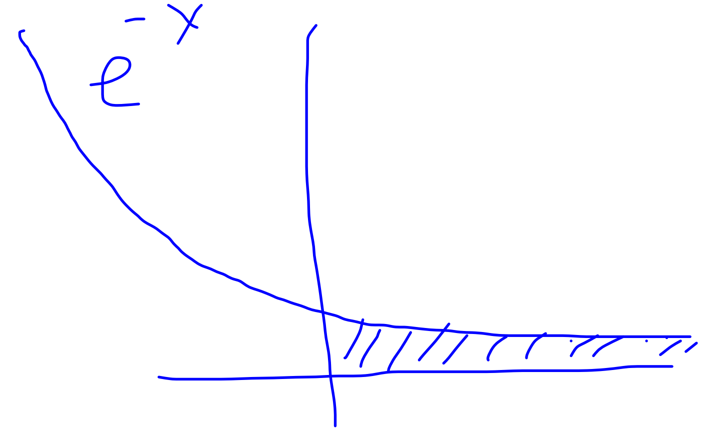
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \ln|x| \Big|_1^t = \lim_{t\to\infty} [\ln t]$$

$$= \infty$$

divergent

Example: Find $\int_0^\infty e^{-x}\,dx$



$$= \lim_{t\to\infty} \int_0^t e^{-x}\,dx$$

$$= \lim_{t\to\infty} \left[-e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty} \left[-e^{-t} - (-e^{-0})\right]$$

$$= \lim_{t\to\infty} \left[1 - \underbrace{e^{-t}}_{0}\right] = 1 \Rightarrow \text{convergent}$$

<u>Example</u>: Evaluate $\int_1^\infty \frac{1}{x^3}\,dx$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{x^{-2}}{2}\right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\underbrace{-\frac{1}{2t^2}}_{\to 0} + \frac{1}{2\cdot 1^2}\right] = \frac{1}{2}$$

$\Rightarrow$ convergent

Example: Calculate $\int_2^{\infty} \frac{1}{x\ln x}\,dx$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \Big[\ln|u|\Big]_{\ln 2}^{\ln t}$$

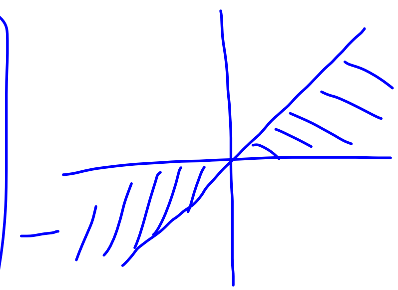
$$= \lim_{t\to\infty} \Big[\underbrace{\ln(\ln t)}_{\to\infty} - \ln(\ln 2)\Big] = \infty$$

divergent

Example: Find $\int_{-\infty}^{\infty} \cos \pi x \, dx$ $= 2 \int_0^{\infty} \cos \pi x \, dx$

even

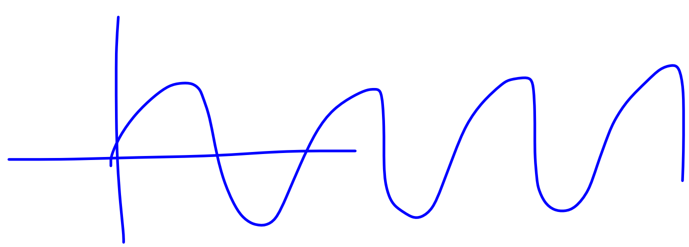
$$\int_{-\infty}^{\infty} x \, dx = \lim_{t \to \infty} \frac{x^2}{2} \Big|_{-t}^{t} = 0$$



$$= 2 \lim_{t \to \infty} \int_0^t \cos \pi x \, dx$$

$$= 2 \lim \left[\frac{1}{\pi} \sin \pi x \right]_0^t$$

$$= \frac{2}{\pi} \lim_{t \to \infty} \sin \pi t \quad \text{dne}$$



divergent

Integrals of Discontinuous Functions

Definition:

- Let f be continuous on $[a, b)$ but discontinuous at b. Then define
$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$
$a \qquad t \to b^-$

  provided that this limit exists.

- Let f be continuous on $(a, b]$ but discontinuous at a. Then define
$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$
$a^+ \leftarrow t \qquad b$

  provided that this limit exists.

  $\int_a^b f(x)dx$ is **convergent** if the limit exists and **divergent** otherwise.

- Let f be discontinuous at c, $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we can write
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example: Find $\int_0^1 \underbrace{\frac{1}{\sqrt{1-x}}}\,dx$

$\rightarrow$ discontinuous at 1

$$= \lim_{t \to 1^-} \int_0^t \frac{1}{\sqrt{1-x}}\,dx$$

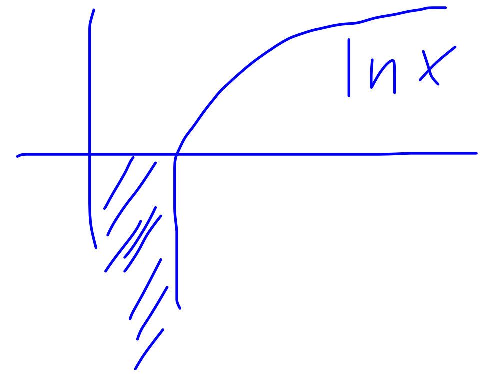
$$u = 1-x$$
$$du = -dx$$

$$= \lim_{t \to 1^-} -\int_1^{1-t} \frac{1}{\sqrt{u}}\,du = \lim_{t \to 1^-} \int_{1-t}^{1} \frac{1}{\sqrt{u}}\,du$$

$$= \lim_{t \to 1^-} \left[2\sqrt{u}\,\right]_{1-t}^{1} = \lim_{t \to 1^-} \left[2\sqrt{1} - 2\sqrt{1-t}\,\right]$$

$$= 2$$

Example: Calculate $\int_0^1 \ln x \, dx$



$$= \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$= \lim_{t \to 0^+} \left[x \ln x \Big|_t^1 - \int_t^1 x \cdot \frac{1}{x} dx \right]$$

$$= \lim_{t \to 0^+} \left[-t \ln t - [1 - t] \right] = -1 \Rightarrow \text{convergent}$$

$$\lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{1/t} \overset{H}{=} \lim_{t \to 0^+} \frac{1/t}{-1/t^2} = \lim_{t \to 0^+} (-t) = 0$$

Example: Find $\int_0^5 \frac{1}{x-2}\,dx$

$$= \int_0^2 \frac{1}{x-2}\,dx + \int_2^5 \frac{1}{x-2}\,dx$$

$$= \lim_{t\to 2^-} \int_0^t \frac{1}{x-2}\,dx + \lim_{t\to 2^+} \int_t^5 \frac{1}{x-2}\,dx$$

$$\lim_{t\to 2^-} \int_0^t \frac{1}{x-2}\,dx = \lim_{t\to 2^-} \Big[\ln|x-2| \Big]_0^t$$

$$= \lim_{t\to 2^-} \Big[\ln|t-2| - \ln 2 \Big] = -\infty$$

$\Rightarrow$ divergent

Example: Evaluate $\int_0^1 \frac{e^{1/x}}{x^3}\,dx$

$$= \lim_{t\to 0^+} \int_t^1 \frac{e^{1/x}}{x^3}\,dx = \lim \int e^{1/x} \cdot \frac{1}{x} \cdot \underbrace{\frac{1}{x^2}dx}_{-du}$$

$u = \frac{1}{x}$

$du = -\frac{1}{x^2}dx$

$$= \lim_{t\to 0^+} -\int_{1/t}^{1} u e^u \, du$$

$w = u \qquad e^u du = dv$

$dw = du \qquad v = e^u$

$$= -\lim_{t\to 0^+}\left[u e^u \Big|_{1/t}^{1} - \int_{1/t}^{1} e^u \, du \right]$$

$$= -\lim_{t\to 0^+}\left[e - \frac{e^{1/t}}{t} - e^1 + e^{1/t} \right] = -\lim_{t\to 0^+} e^{1/t}\left[1 - \frac{1}{t}\right]$$

$s = \frac{1}{t}$

$$= -\lim_{s\to\infty} e^s (1 - s) = +\infty \Rightarrow \text{divergent}$$

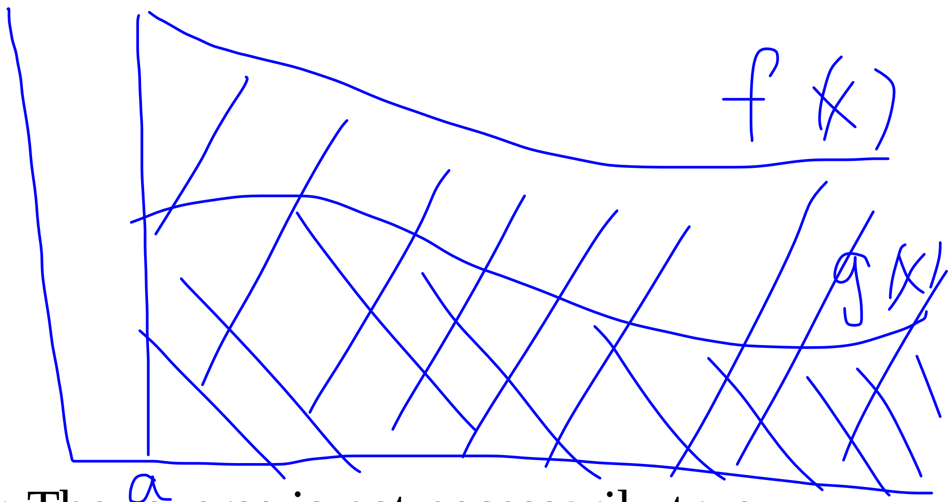
Comparison Test

Very often we would like to know if the integral is convergent or not even though we cannot find its value.

Let f and g be continuous functions defined on $[a,\infty)$ and $0 \le g(x) \le f(x)$.

Then

- If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is also convergent.
- If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is also divergent.



Note: The reverse is not necessarily true.

Example: Investigate the convergence of the improper integral

$$\int_0^\infty e^{-x^2}dx \implies \text{convergent}$$

$$= \underbrace{\int_0^1 e^{-x^2}dx}_{\text{convergent}} + \int_1^\infty e^{-x^2}dx$$

$$x \geq 1$$

$$x^2 \geq x$$

$$-x^2 \leq -x$$

$$\underset{g(x)}{e^{-x^2}} \leq \underset{f(x)}{e^{-x}}$$

$$\int_1^\infty e^{-x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx = \lim_{t\to\infty} -e^{-x}\Big|_1^t$$

$$= \lim_{t\to\infty}\left[e^{-1} - \underset{\searrow 0}{e^{-t}}\right] = e^{-1} \implies \text{convergent}$$

$$\implies \int_1^\infty e^{-x^2}dx \text{ is also convergent}$$

Example: Investigate the convergence of the improper integral

$$\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$$

$$0 \le \underbrace{\frac{\sin^2 x}{\sqrt{x}}}_{g(x)} \le \underbrace{\frac{1}{\sqrt{x}}}_{f(x)}$$

$$\int_0^{\pi} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^{\pi} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 2\sqrt{x}\,\Big|_t^{\pi}$$

$$= \lim_{t\to 0^+} \left[2\sqrt{\pi} - 2\sqrt{t}\right] = 2\sqrt{\pi}$$

convergent

Thus, $\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$ is also convergent

Example: Investigate the convergence of the improper integral

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx$$

$$x \geq 1 \Rightarrow x^3 \geq 1$$

$$x^3+5 \geq x^3$$

$$\sqrt{x^3+5} \geq \sqrt{x^3}$$

$$\frac{1}{\sqrt{x^3+5}} \leq \frac{1}{\sqrt{x^3}}$$

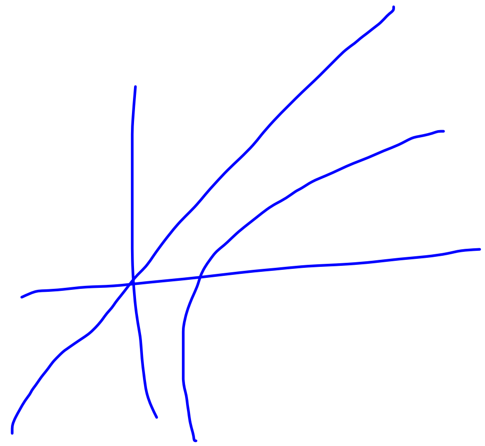
$$\int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x^3}}\,dx = \lim_{t\to\infty} \left[-2x^{-\frac{1}{2}}\right]_1^t$$

$$= 2\lim_{t\to\infty}\Big[1 - \underbrace{\frac{1}{\sqrt{t}}}_{\to 0}\Big] = 2 \Rightarrow \text{convergent}$$

$$\Rightarrow \int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx \text{ is also convergent}$$

Example: Investigate the convergence of the improper integral

$$\int_4^\infty \frac{1}{\ln x - 1}\,dx$$



$$\ln x - 1 < \ln x < x$$

$$\frac{1}{\ln x - 1} > \frac{1}{x}$$

$$\int_4^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_4^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x \Big|_4^t$$

$$= \lim_{t\to\infty} [\ln t - \ln 4] = \infty \Rightarrow \text{divergent}$$

$$\Rightarrow \int_4^\infty \frac{1}{\ln x - 1}\,dx \text{ is also divergent}$$

Example: Investigate the convergence of the improper integral

$$\int_3^\infty \frac{1}{x+e^x}\,dx$$

$$x+e^x > x \implies \frac{1}{x+e^x} < \frac{1}{x}$$

$$\int_3^\infty \frac{1}{x}\,dx = \infty \implies \text{divergent}$$

$$\not\Rightarrow \int_3^\infty \frac{1}{x+e^x}\,dx \text{ is divergent}$$

Another attempt: $x+e^x > e^x \implies \frac{1}{x+e^x} < \frac{1}{e^x}$

$$\int_3^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_3^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_3^t$$

$$= \lim_{t\to\infty}\left[e^{-3} - \underbrace{e^{-t}}_{\to 0}\right] = e^{-3} \implies \text{convergent}$$

Thus, $\int_3^\infty \frac{1}{x+e^x}\,dx$ is convergent

<u>Example</u>: Investigate the convergence of the improper integral

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx$$

$$x - e^{-x} < x$$

$$\frac{1}{x - e^{-x}} > \frac{1}{x}$$

$$\int_3^\infty \frac{1}{x}\,dx = \infty \Rightarrow \text{divergent}$$

$$\Rightarrow \int_3^\infty \frac{1}{x - e^{-x}}\,dx \text{ is also divergent}$$