<u>Lecture 1</u> (Review of High School Math: Functions and Models)

Introduction: Numbers and their properties

Addition:

(1) (Associative law) If a, b, and c are any numbers, then

$$a + (b + c) = (a + b) + c$$

(2) (Existence of an additive identity) If a is any number, then

$$a + 0 = 0 + a = a$$

(3) (Existence of additive inverses) For every number a, there is a number $-a$ such that

$$a + (-a) = (-a) + a = 0$$

(4) (Commutative law) If a and b are any numbers, then

$$a + b = b + a$$

Multiplication:

(5) (Associative law) If a, b, and c are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(6) (Existence of an multiplicative identity) If a is any number, then

$$a \cdot 1 = 1 \cdot a = a$$

(7) (Existence of multiplicative inverses) For every number $a \neq 0$, there is a number a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

(Note: division by 0 is *always* undefined!)

(8) (Commutative law) If a and b are any numbers, then

$$a \cdot b = b \cdot a$$

(9) (Distributive law) If a, b, and c are any numbers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Definition: The numbers a satisfying $a > 0$ are called **positive**, while those numbers a satisfying $a < 0$ are called **negative**.

For any number a, we define the **absolute value** $|a|$ of a as follows:

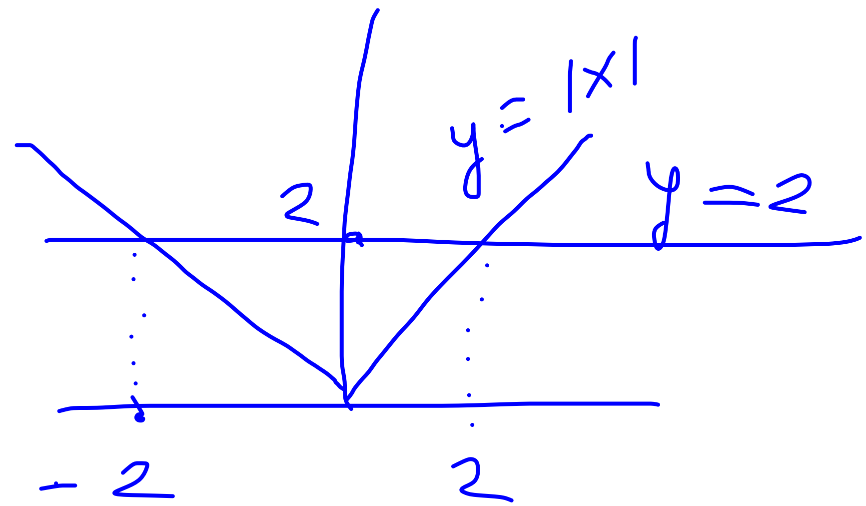
$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$$

Note: $|a|$ is always positive, except when $a = 0$

Example:

$$|-2| = 2$$

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$



$$|x| \leq 2$$

$$-2 \leq x \leq 2$$

Theorem (Triangle Inequality): For all numbers a and b, we have

$$|a+b| \le |a| + |b|$$

Proof:

Note: $a \le |a|$

$$\sqrt{|a+b|^2} = (a+b)^2 = a^2 + 2ab + b^2$$

$$= |a|^2 + 2ab + |b|^2$$

$$\le |a|^2 + 2|a||b| + |b|^2$$

$$= \sqrt{(|a| + |b|)^2}$$

$$|a+b| \le |a| + |b|$$

$\blacksquare$

Exercises

1. Prove the following:

(a) $x^2 - y^2 = (x - y)(x + y)$

(b) $(x \pm y)^2 = x^2 \pm 2xy + y^2$

(c) $x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$

2. What is wrong with the following «proof»?

Let

$$x = y$$

then

$$x^2 = xy$$

$$x^2 - y^2 = xy - y^2$$

$$(x + y)(x - y) = y(x - y)$$

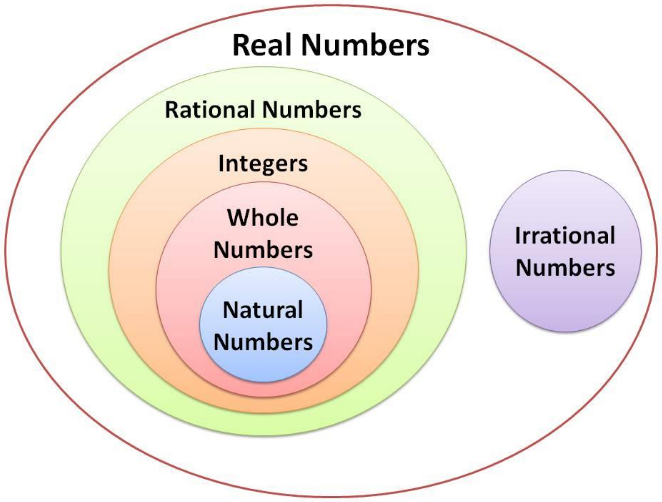
$$x + y = y$$

$$2y = y$$

$$2 = 1$$

wrong!
$x - y = 0$
cannot
divide by 0

What types of numbers are there?...



The simplest numbers are the «counting numbers»:

$$1, 2, 3, \ldots$$

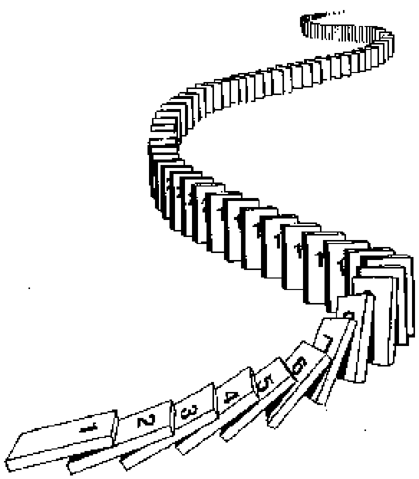
We call them **natural numbers** and denote by $\mathbb{N}$.

The most basic property of $\mathbb{N}$ is the principle of «*mathematical induction*».

<u>Mathematical Induction</u>: Suppose P(n) means that the property P holds for the number n. Then P(n) is true for all natural numbers n provided that

(1) P(1) is true
(2) Whenever P(k) is true, P(k + 1) is true.

A standard analogy is a string of dominoes which are arranged in such a way that if any given domino is knocked over then it in turn knocks over the next one.



This analogy is a good one but it is only an analogy, and we have to remember that in the domino situation there is only a **finite number** of dominoes.

Example: Show that $1 + \cdots + n = \frac{n(n+1)}{2}$ (*)

Solution:

(1) Show (*) is true for $n=1$

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1$$

(2) Assume (*) is true for $n=k$, i.e.

$$1 + \ldots + k = \frac{k(k+1)}{2}$$

Need to prove for $n=k+1$, i.e.

$$\underbrace{1 + \ldots + k}_{\frac{k(k+1)}{2}} + (k+1) \overset{?}{=} \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$\boxed{3}$

Exercise

Prove by induction on n that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$ (note that if $r = 1$, you can easily calculate the sum)

Other numbers:

Integers: ..., -2, -1, 0, 1, 2,.... This set is denoted by $\mathbb{Z}$.

Rational numbers: $\frac{m}{n}$, $n \neq 0$, $m, n \epsilon \mathbb{Z}$. This set is denoted by $\mathbb{Q}$.

Real numbers: denoted by $\mathbb{R}$.

Real numbers include rational and **irrational numbers** (e.g. π or $\sqrt{2}$, i.e. numbers that can be represented by infinite decimals).

Why is $\sqrt{2}$ irrational? Assume it is rational.

$\sqrt{2} = \frac{a}{b} \rightarrow$ irreducible

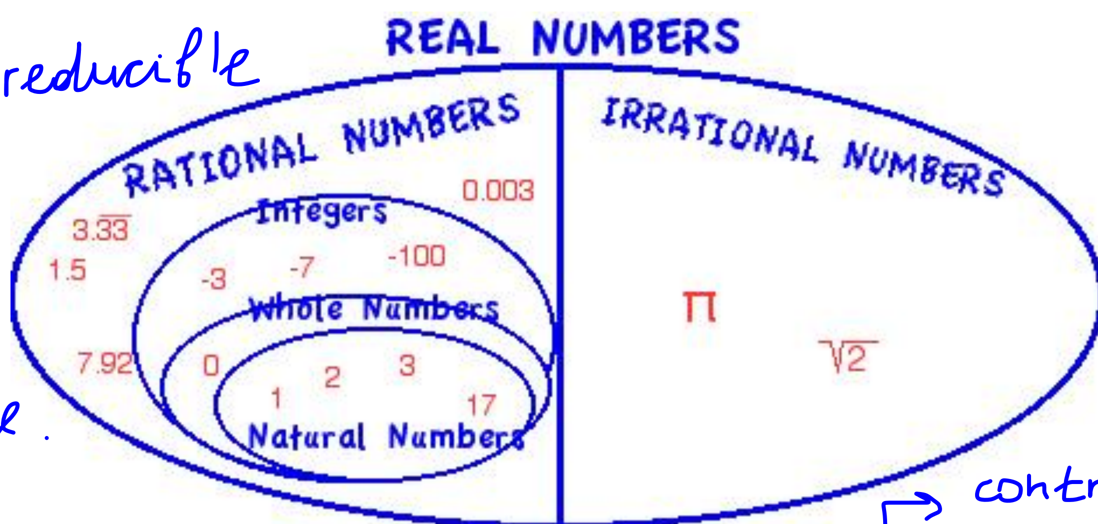
$\sqrt{2}\, b = a$

$2b^2 = a^2$

So a is even, i.e.

$a = 2k,\ k \in \mathbb{Z}$

$2b^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b$ is even, i.e. $b = 2m$ $\rightarrow$ contradiction to irreducibility of $\frac{a}{b}$



Set notation and set operations

Definition: A **set** A is a collection of objects which are called elements or members.

Example: $A = \{-1, 0, 1, 2\}$

Symbols that we shall use:

$x \in A$ (x belongs to A)

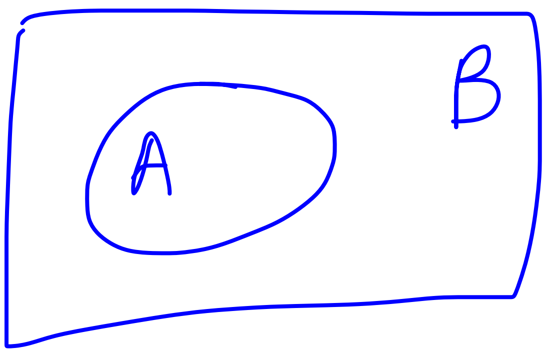
$-1 \in A$

$x \notin A$ (x does not belong to A)

$5 \notin A$

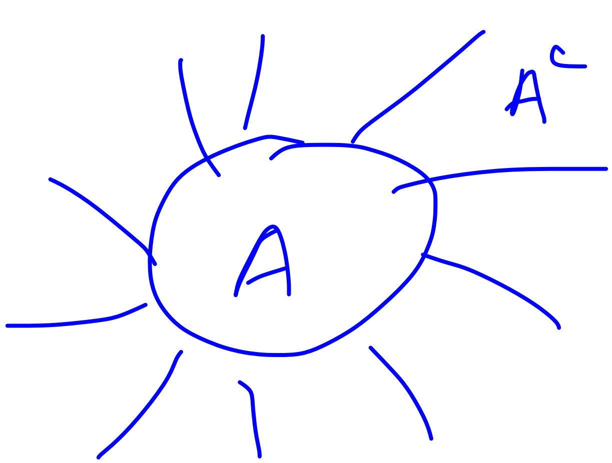
Subset: $A \subset B$ any $x \in A \Rightarrow x \in B$

Venn Diagram:



$A \subset B$

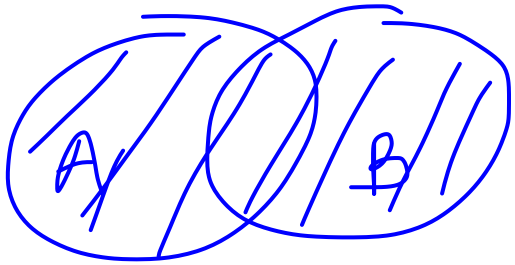
Complement: A^c $x \in A^c \Rightarrow x \notin A$



$A = \{-1, 0, 1, 2\}$

$A^c = \mathbb{R} \setminus \{-1, 0, 1, 2\}$

Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$

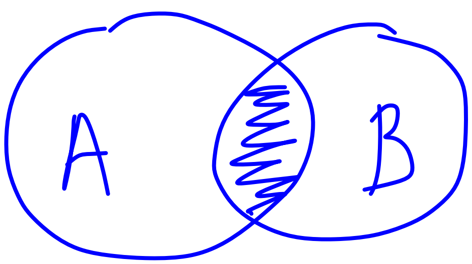


$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

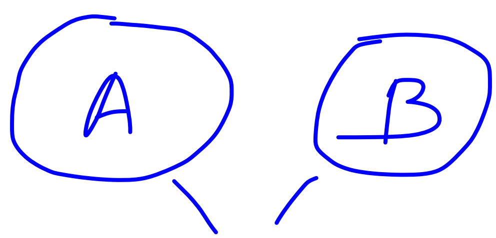
$A \cup B = \{1, 2, 3, 4, 5, 6\}$

Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$



$A \cap B = \{3, 4\}$

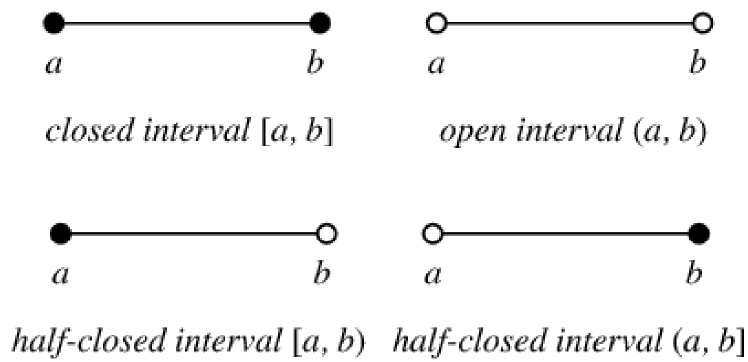
Empty set: $\emptyset$



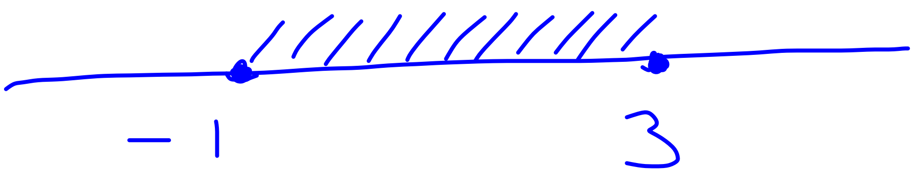
disjoint

$A \cap B = \emptyset$

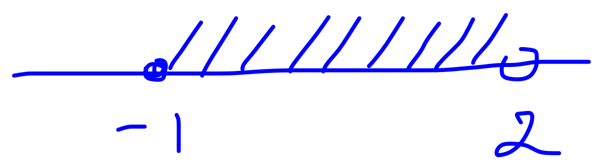
Intervals: $[a,b]$, (a,b), $[a,b)$, $(a,b]$



Ex. $x \in [-1, 3]$

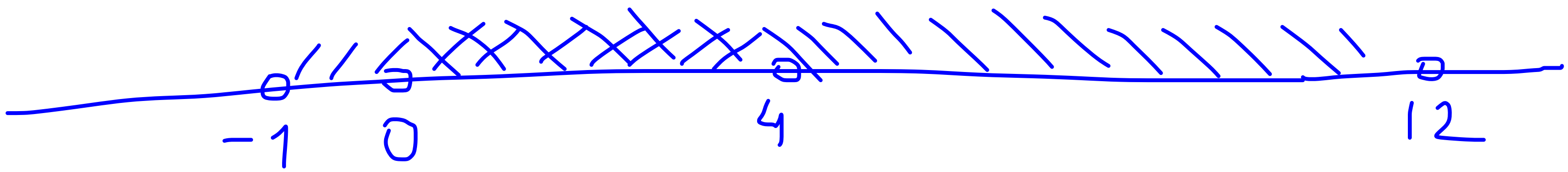


$x \in [-1, 2)$

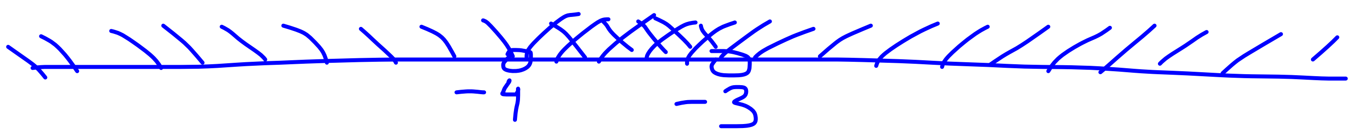


Example:

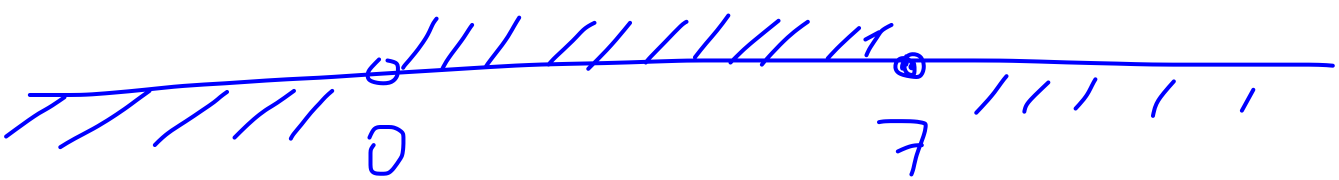
$(-1, 4) \cap (0, 12) =$ $(0, 4)$



$(-\infty, -3) \cup (-4, \infty) =$ $\mathbb{R} = (-\infty, \infty)$



$(0, 7]^c =$ $(-\infty, 0] \cup (7, \infty)$



Solving inequalities

<u>Example</u>: Solve $2 - 3x > 8$.

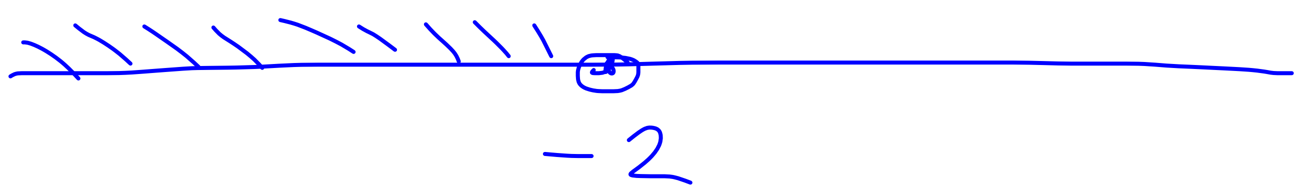
$$2 - 8 > 3x$$

$$3x < -6$$

$$x < -2$$

Express the answer as an interval and graphically.

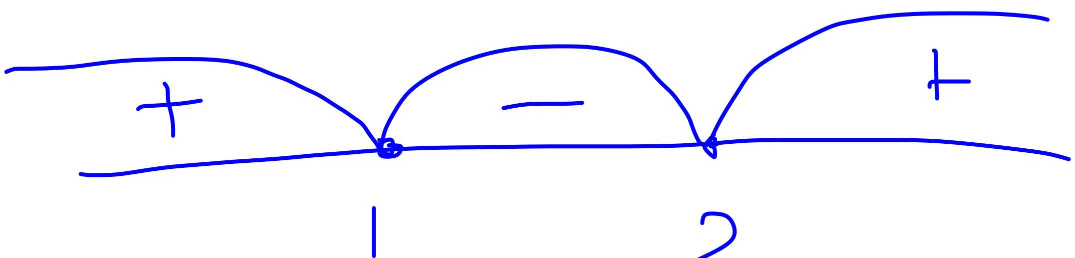
$$(-\infty, -2)$$



Example: $x^2 - 3x + 3 \geq 1$

$$x^2 - 3x + 2 \geq 0$$

$$(x-2)(x-1) \geq 0$$



$x - 2 < 0$ $\quad$ $x - 2 < 0$ $\quad$ $x - 2 > 0$

$x - 1 < 0$ $\quad$ $x - 1 > 0$ $\quad$ $x - 1 > 0$

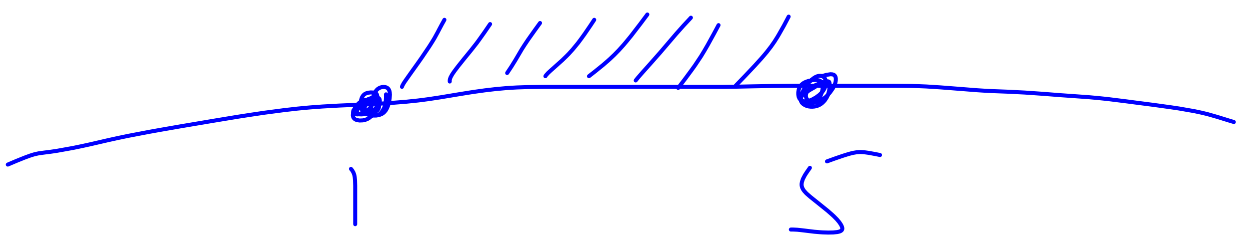
$$x \in (-\infty, 1] \cup [2, \infty)$$

Example: Solve $|x-3| \leq 2$

$$-2 \leq x-3 \leq 2 \qquad +3$$

$$1 \leq x \leq 5$$

$$x \in [1, 5]$$

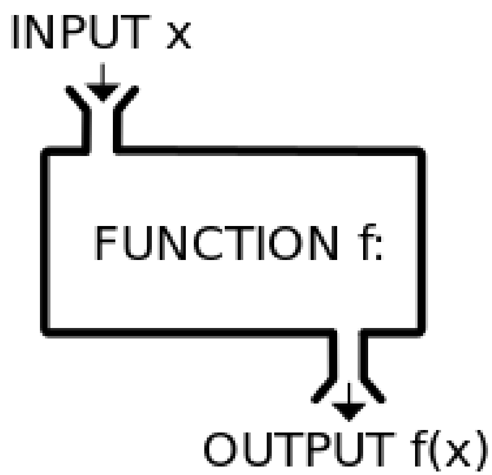


Functions

What is a function?

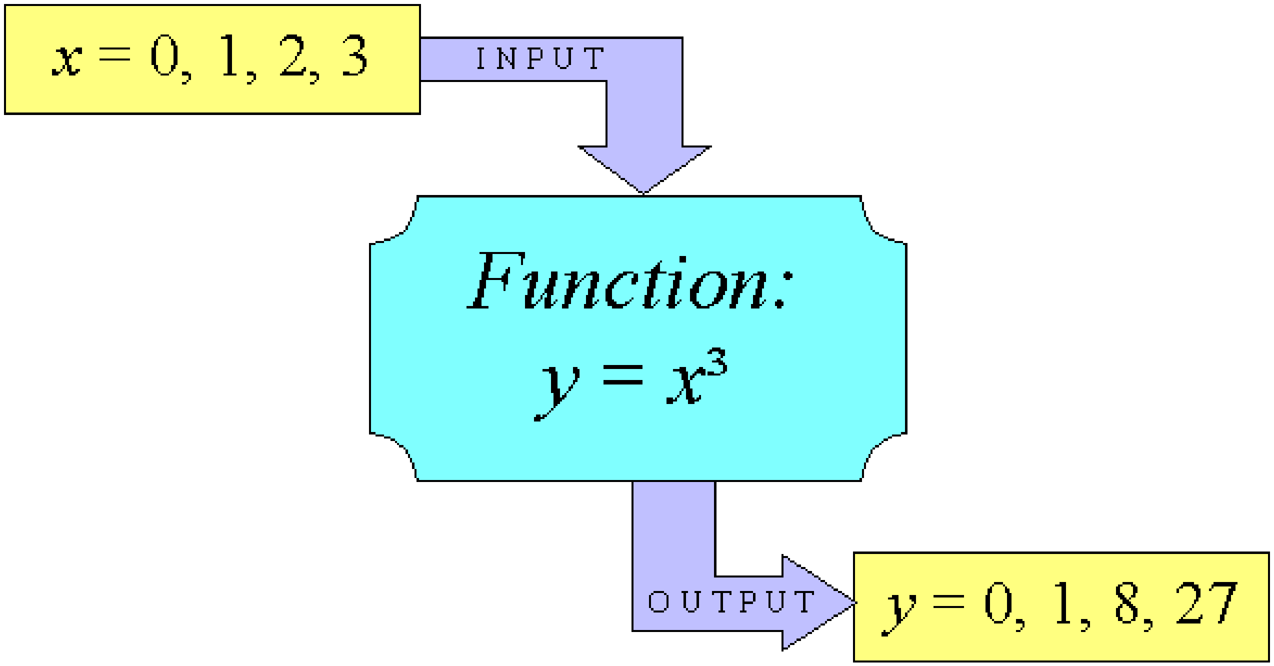
- A **function** is a rule which assigns, to each of certain real numbers, some other real number.

Notation: $f(x)$.



Example: The rule which assigns to each number the cube of that number:

$$f(x) = x^3$$



Using notations:

- A **function** f is a rule that assigns to each element x from some set D exactly one element, $f(x)$, in a set E.
- D is a set of real numbers, called the **domain** of the function.
- E is a set of real numbers, called the **range** of the function, it is the set of all possible values of $f(x)$ defined for every x in the domain.
- We call x an **independent** variable, and $y = f(x)$ a **dependent** variable.

Examples: Find domain and range in interval notation.

(1) $f(x) = x^2$

$D = \mathbb{R} = (-\infty, \infty)$

$E = [0, \infty)$

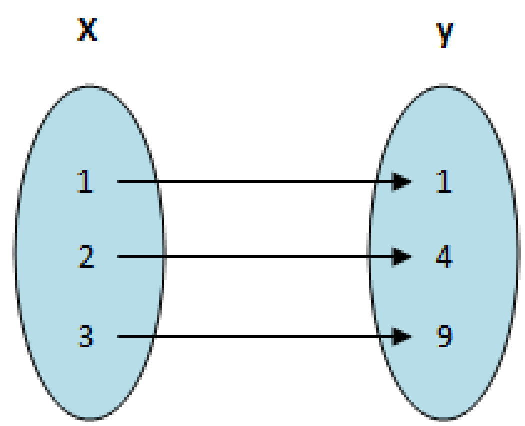
(2) $f(x) = \frac{1}{x-1}$

$D = \{x \in \mathbb{R} : x \neq 1\} = (-\infty, 1) \cup (1, \infty)$

$E = (-\infty, 0) \cup (0, \infty)$

Visualizing a function

There are different ways to picture a function. One of them is an **arrow diagram**:

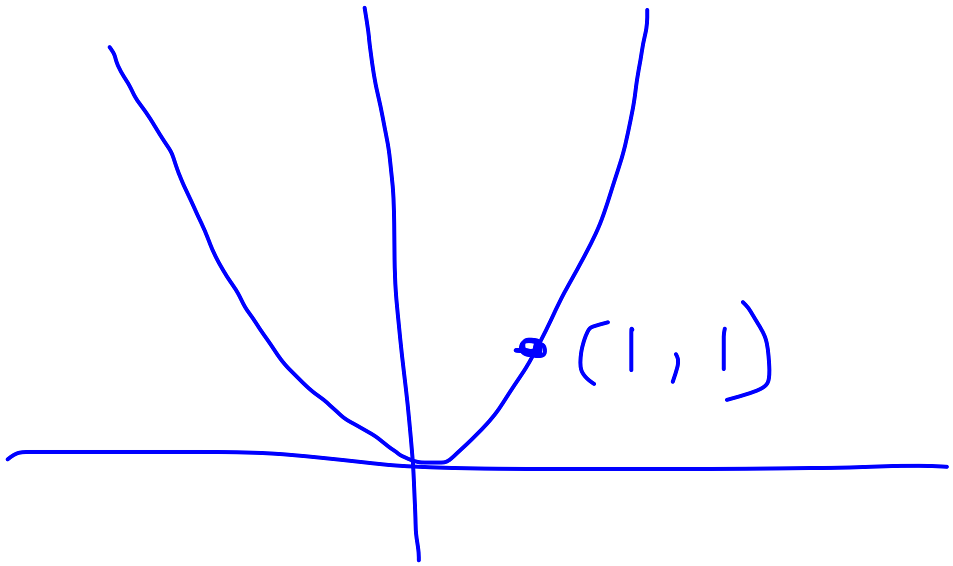


Each arrow connects an element of D to an element of E.

The most common way to picture a function is by drawing a graph.

Definition: A **graph** is the set of ordered pairs $\{(x, f(x)) | x \epsilon D\}$.

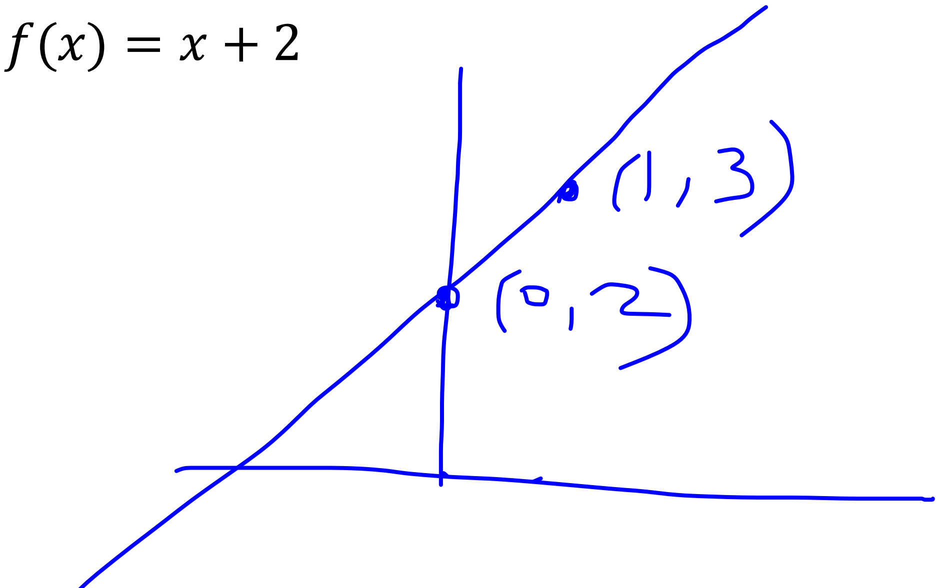
$y = x^2$



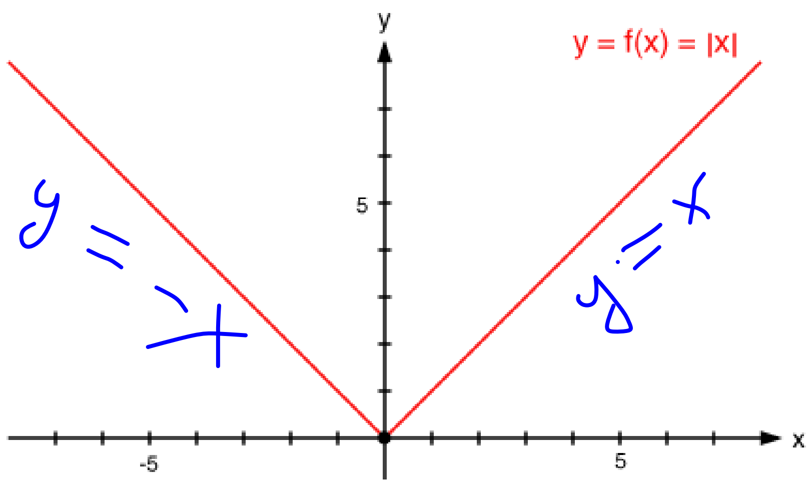
Example: Given $f(x) = x^2 - 2x + 1$, find $f(6)$.

$f(6) = 6^2 - 2 \cdot 6 + 1 = 25$

Example: Graph $f(x) = x + 2$

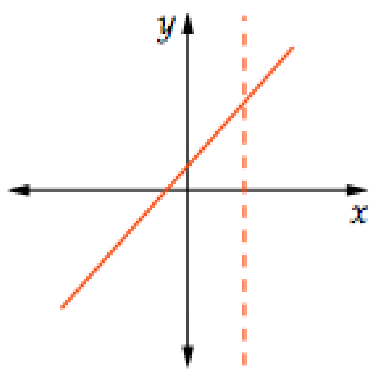


Example: Graph $f(x) = |x|$

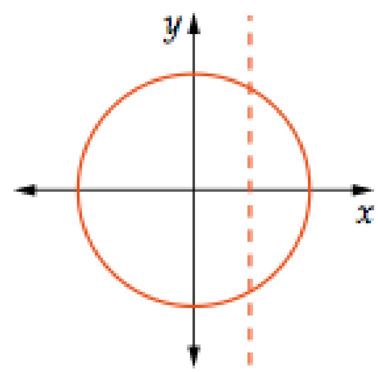


When you look at the graph, how do you know you are looking at a function?

<u>Vertical Line Test</u>: A curve in the *xy*-plane is the graph of a function of *x* if and only if no vertical line intersects the curve more than once.



Cuts once, so graph represents a function.



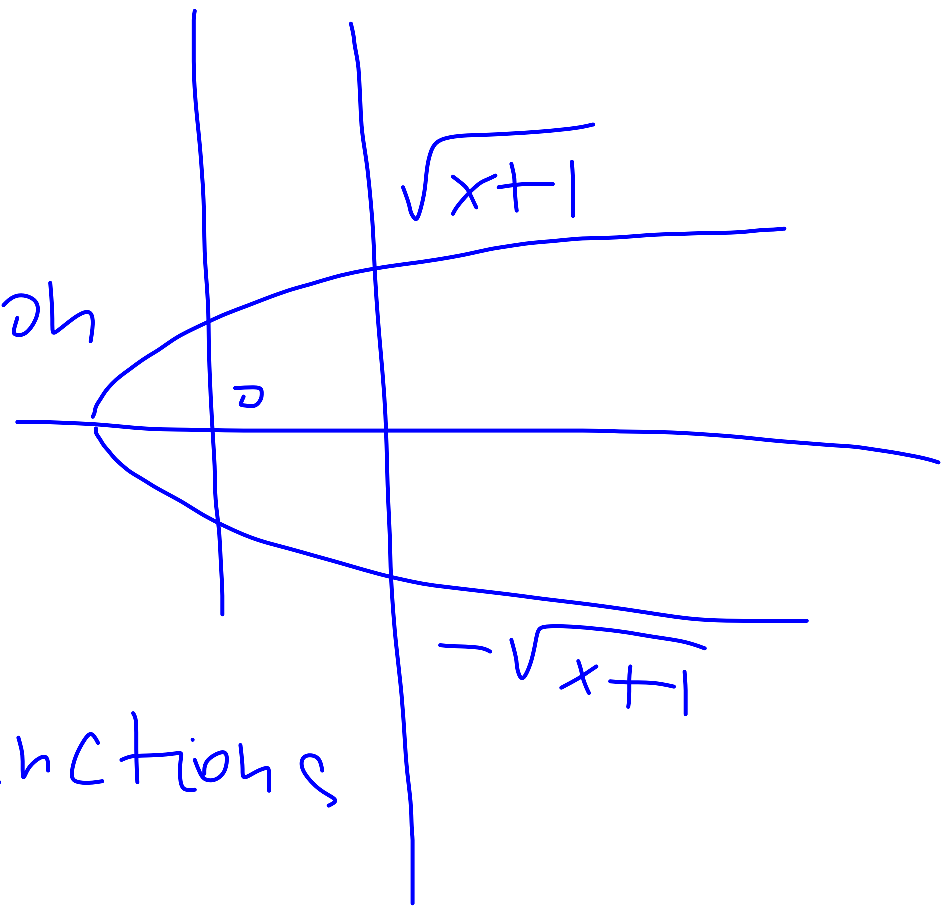
Cuts twice, so graph does not represent a function.

Example: $x = y^2 - 1$

$y^2 = x + 1$

$y = \pm\sqrt{x+1}$

not a function



$y = \sqrt{x+1}$

or

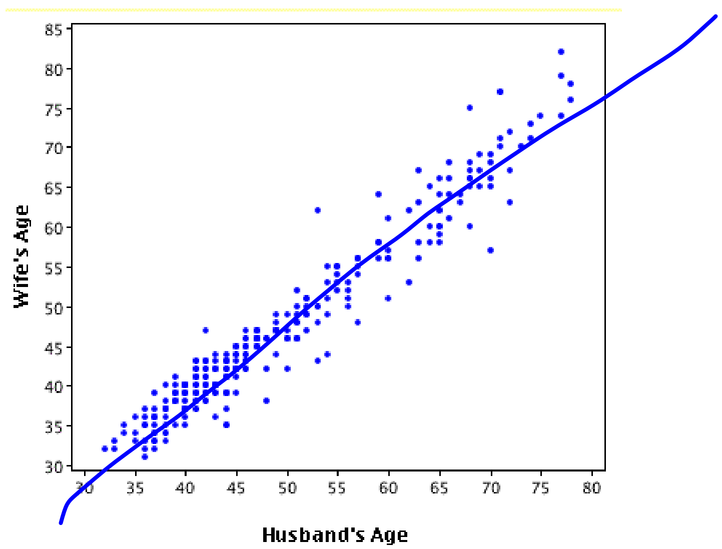
$y = -\sqrt{x+1}$

functions

Mathematical models: What kind of functions are there?

A mathematical model is a mathematical description (function or equation) of a real-world phenomenon.

<u>Example</u>: There is a strong positive linear relationship between husband's age and wife's age.



We can use a *linear model* to describe this relationship!

Definition: We say y is a **linear function** of x if $y = f(x) = mx + b$

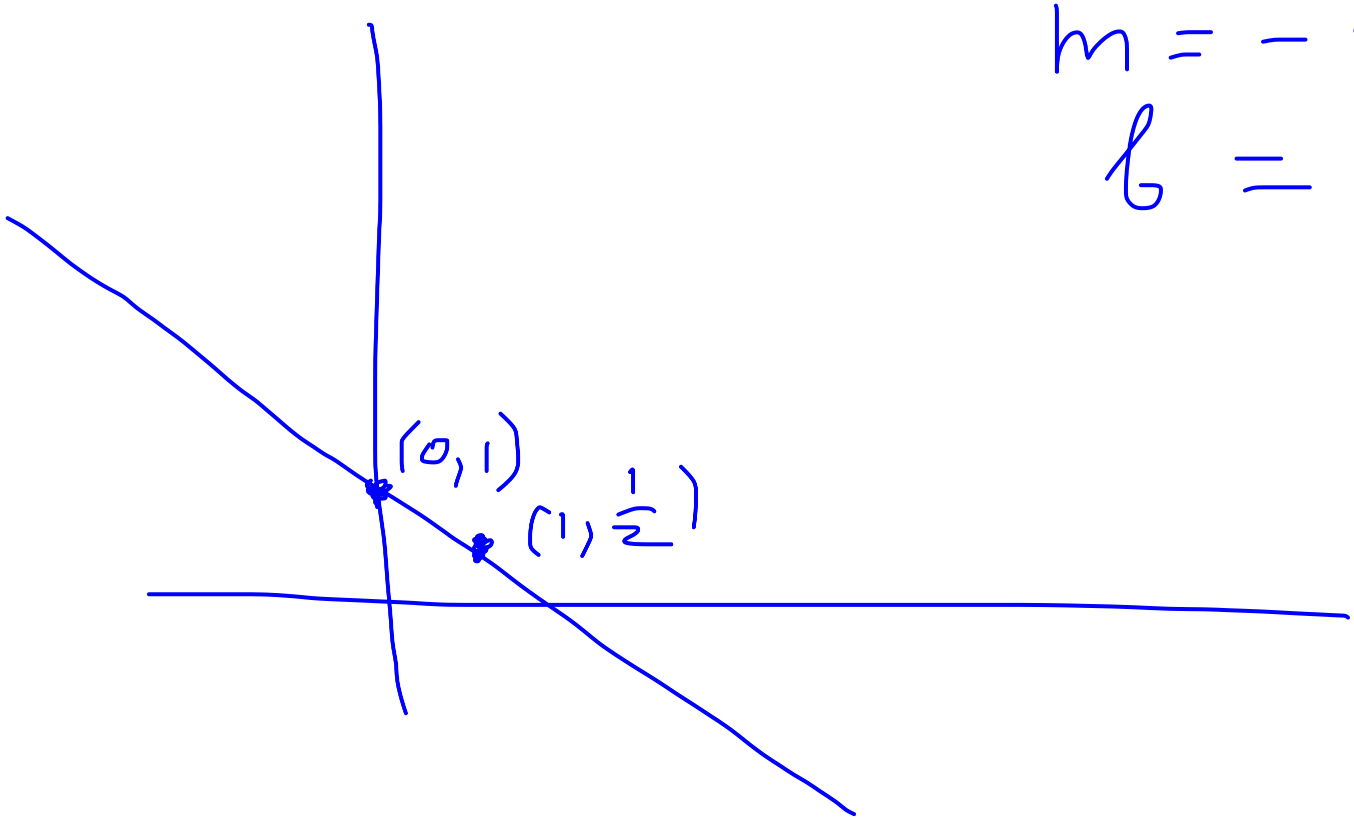
– equation of a line, where

- m is the **slope** of the line, the amount by which y changes when x increases by one unit.
- b is the y-**intercept**, the value of y when $x = 0$.

Example: $y = -0.5x + 1$

$m = -0.5$

$b = 1$

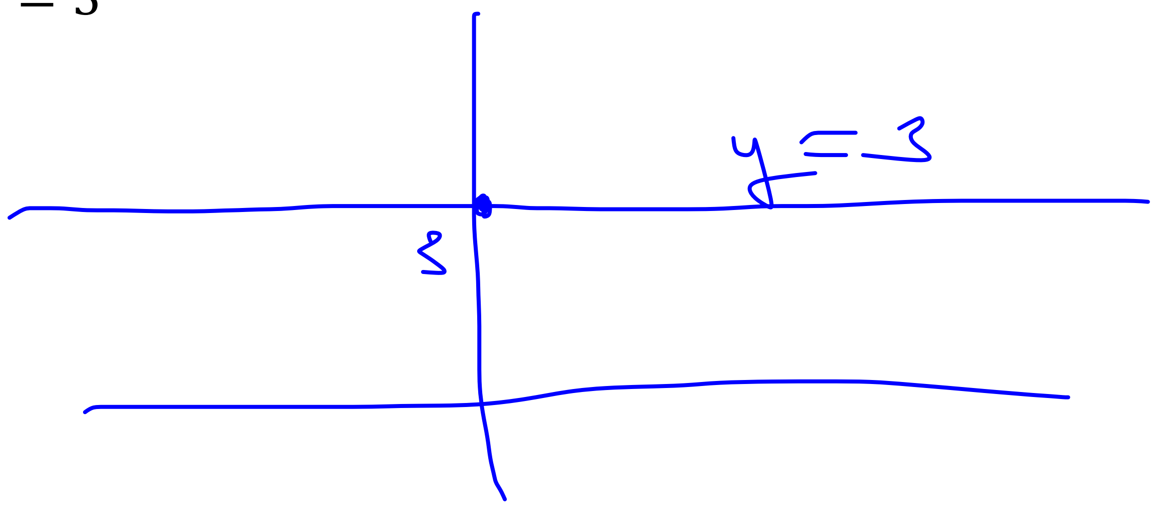


Definition: A function f is a **polynomial function** if there are real numbers $a_0, a_1, \ldots, a_n$ such that $P(x) = f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, for all x, n is a nonnegative integer.

The numbers $a_0, a_1, \ldots, a_n$ are called **coefficients** of the polynomial. The highest power of x with a nonzero coefficient is called the **degree** of the polynomial.

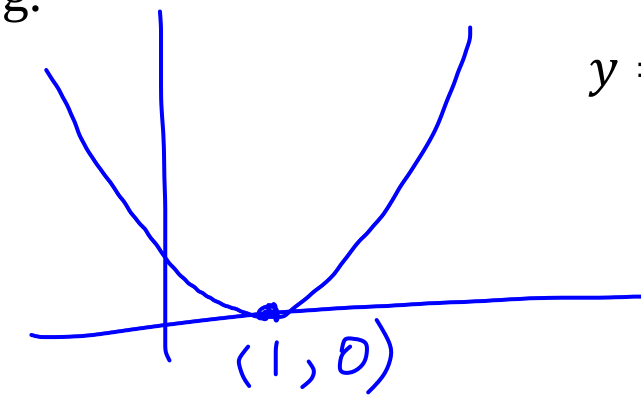
Examples:

1) A polynomial of degree 0 is a constant function $f(x) = c$
   e.g. $y = 3$



2) A polynomial of degree 1 is a linear function $f(x) = mx + b$.

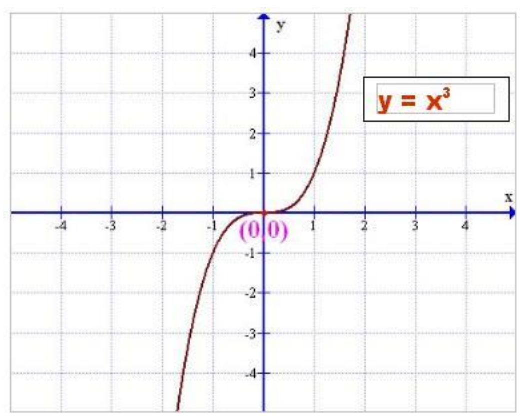
3) A polynomial of degree 2 is a quadratic function $f(x) = ax^2 + bx + c$, e.g.



$$y = x^2 - 2x + 1 = (x-1)^2$$

The graph is called a *parabola*.

4) A polynomial of degree 3 is a cubic function $f(x) = ax^3 + bx^2 + cx + d$, e.g. $y = x^3$

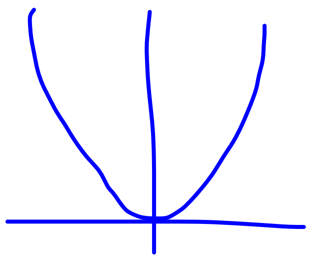


Definition: If $f(-x) = f(x)$ for every $x \epsilon D$, then f is called an **even function**. If $f(-x) = -f(x)$ for every $x \epsilon D$, then f is called an **odd function**.

Example:

$f(x) = x^2$ is an even polynomial function.

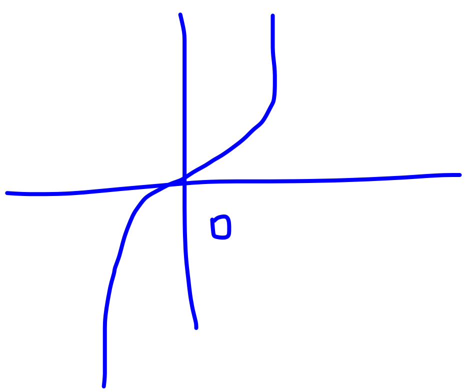
$$f(-x) = (-x)^2 = x^2 = f(x)$$



The graph of an even function is symmetric with respect to the y-axis.

$f(x) = x^3$ is an odd polynomial function.

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$



The graph of an odd function is symmetric about the origin.

What about $f(x) = x^2 - 2x + 1$?

$$f(-x) = (-x)^2 - 2(-x) + 1$$

$$= x^2 + 2x + 1 \neq f(x)$$

$$\neq -f(x)$$

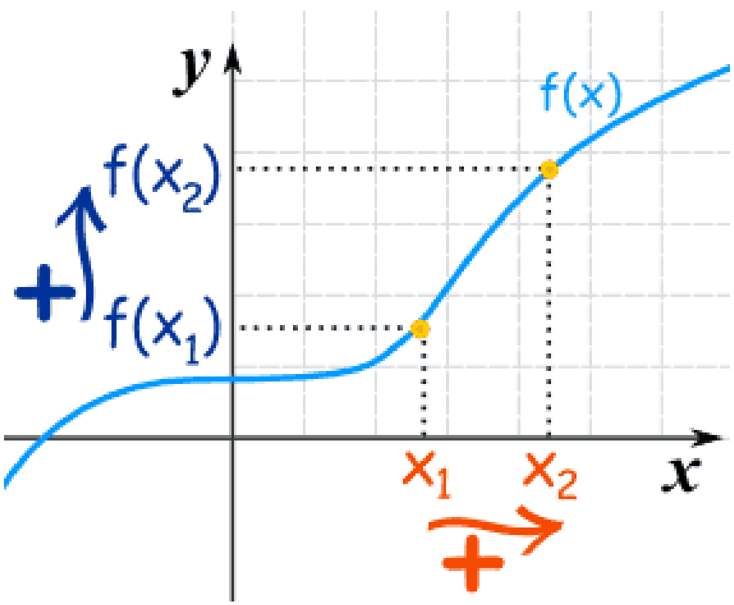
neither odd nor even

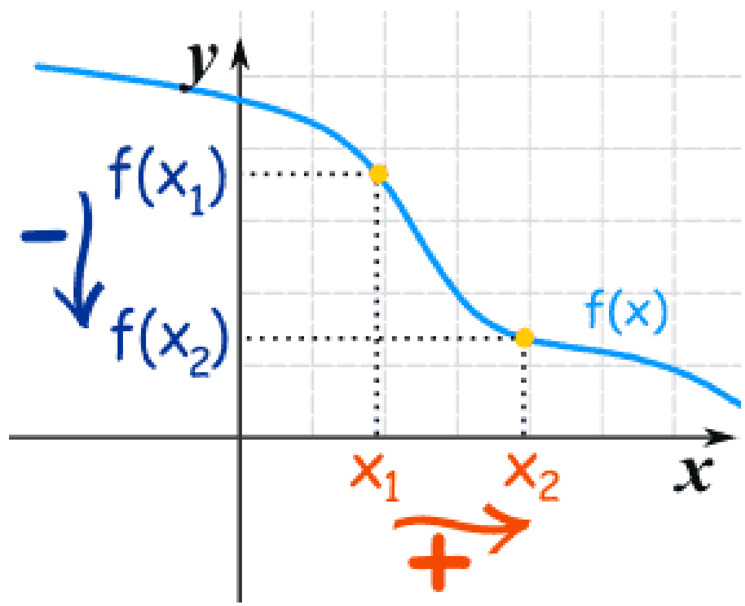
Definition: A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

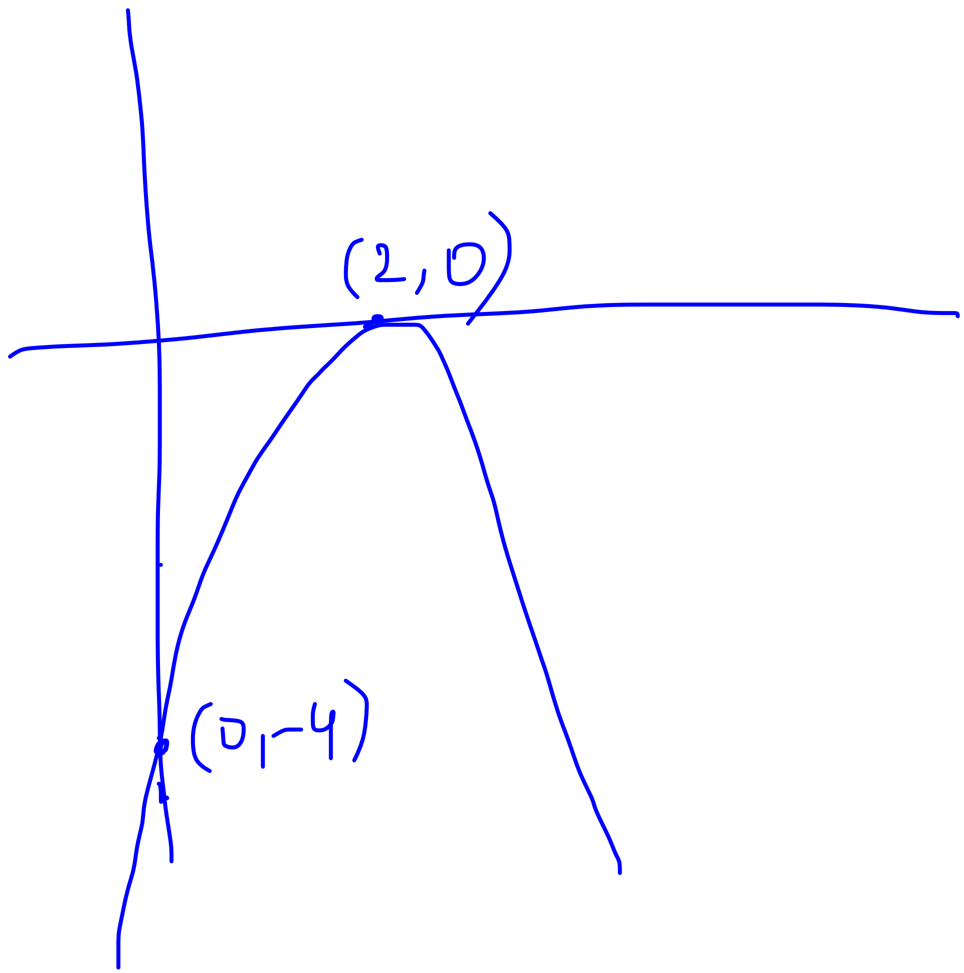
$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I$$





Example: Given $f(x) = -x^2 + 4x - 4$, find the intervals where $f(x)$ is increasing/decreasing.

$$f(x) = -(x^2 - 4x + 4) = -(x-2)^2$$



$f(x) \nearrow$ on $(-\infty, 2)$

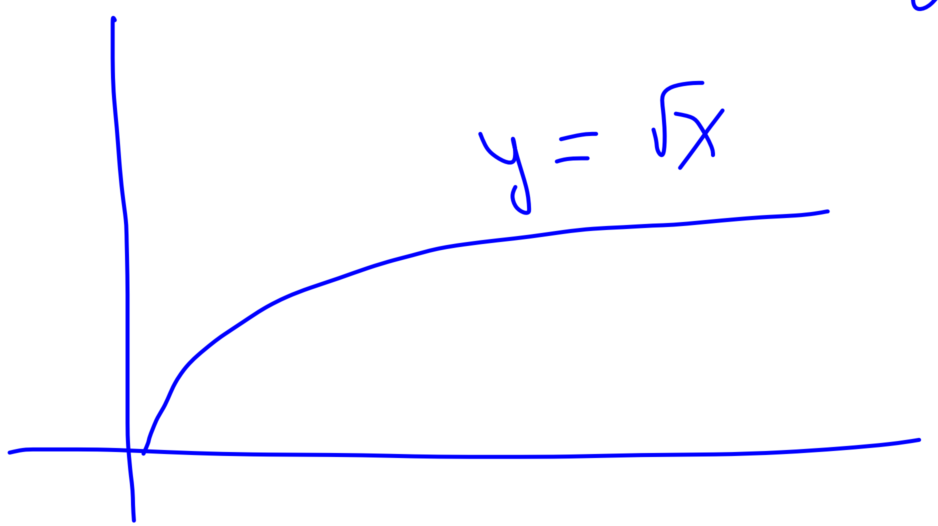
$f(x) \searrow$ on $(2, \infty)$

Definition: A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider the following cases:

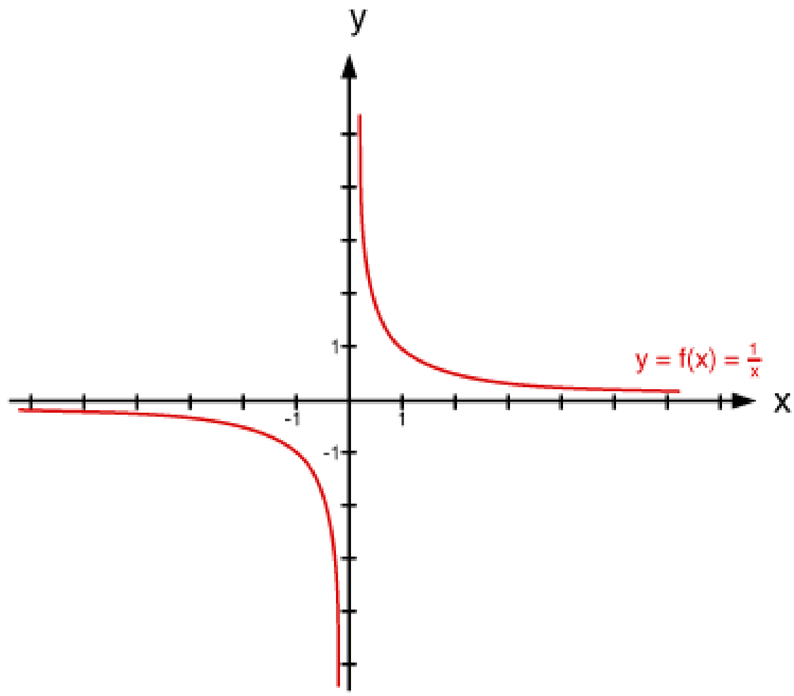
- If $a = n$, where n is a positive integer, then $f(x) = x^n$ is a *polynomial function.*
- If $a = 1/n$, where n is a positive integer, then $f(x) = \sqrt[n]{x}$ is a **root function**.

Example: $y = \sqrt{x}$

$a = \frac{1}{2}$



- If $a = -1$, then $f(x) = x^{-1} = \frac{1}{x}$ is a **reciprocal function**.



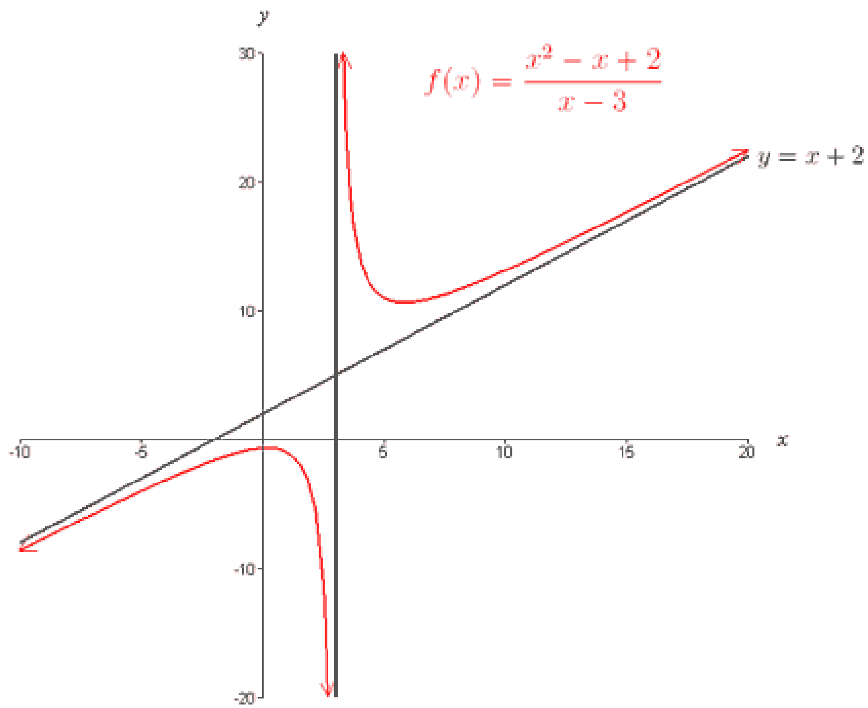
The graph is called a *hyperbola* with the coordinate axes as its asymptotes.

Definition: A function f is called a **rational function**, if it can be written as a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

$Q(x) \neq 0$

Example: $f(x) = \frac{x^2-x+2}{x-3}$



Definition: A function f is called an **algebraic function** if it is constructed by applying algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) to the polynomials.

Examples:

$$f(x) = \sqrt{x^2 + 2}$$

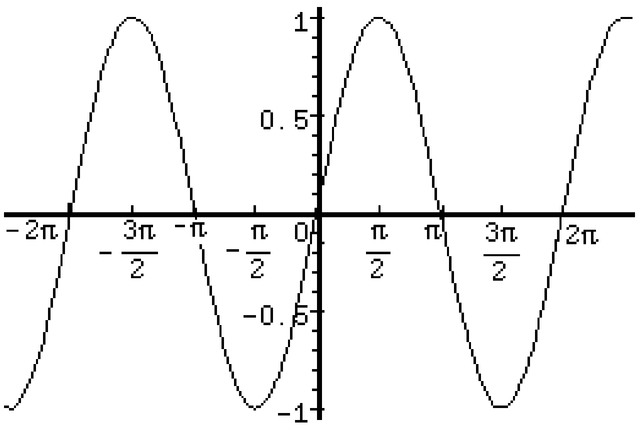
$$f(x) = \frac{1-x}{x^2+1}$$

$$f(x) = \sqrt{x^2 + 2} + \frac{1 - x}{x^2 + 1}$$

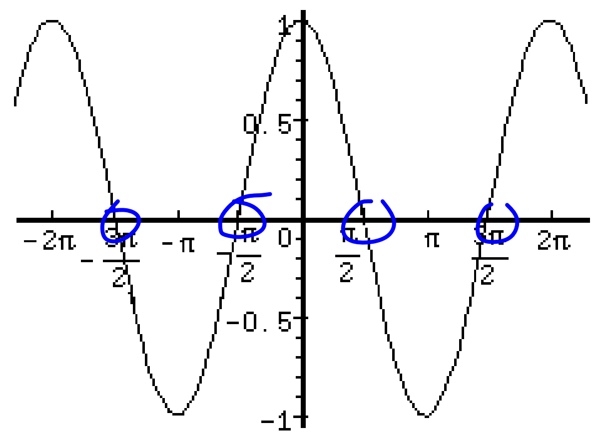
Trigonometric functions (review):

$f(x) = \sin x$

$f(x) = \cos x$



sinx



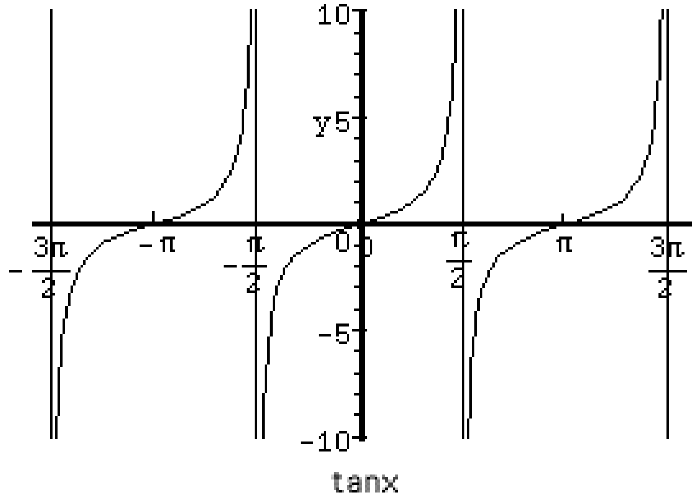
cosx

$D = \mathbb{R}$

$E = [-1, 1]$

period $= 2\pi$

$f(x) = \tan x$ $= \frac{\sin x}{\cos x}$



$D = \{x \in \mathbb{R},\ x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$

$E = \mathbb{R}$ period $= \pi$

The remaining functions: cosecant, secant, and cotangent, are the reciprocal of the ones above.

Partial table of values for trigonometric functions:

Angle θ				
Degrees	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	undefined
180	π	0	-1	0
270	$\frac{3\pi}{2}$	-1	0	undefined
360	2π	0	1	0

Identities

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Reduction Formulas:

$$\sin(-\theta) = -\sin\theta \qquad \sin\theta = -\sin(\theta - \pi)$$
$$\cos(-\theta) = \cos\theta \qquad \cos\theta = -\cos(\theta - \pi)$$
$$\tan(-\theta) = -\tan\theta \qquad \tan\theta = \tan(\theta - \pi)$$

Sum or Difference of Two Angles:

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Half–Angle Formulas:

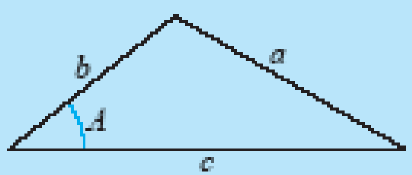
$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$
$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

Double–Angle Formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

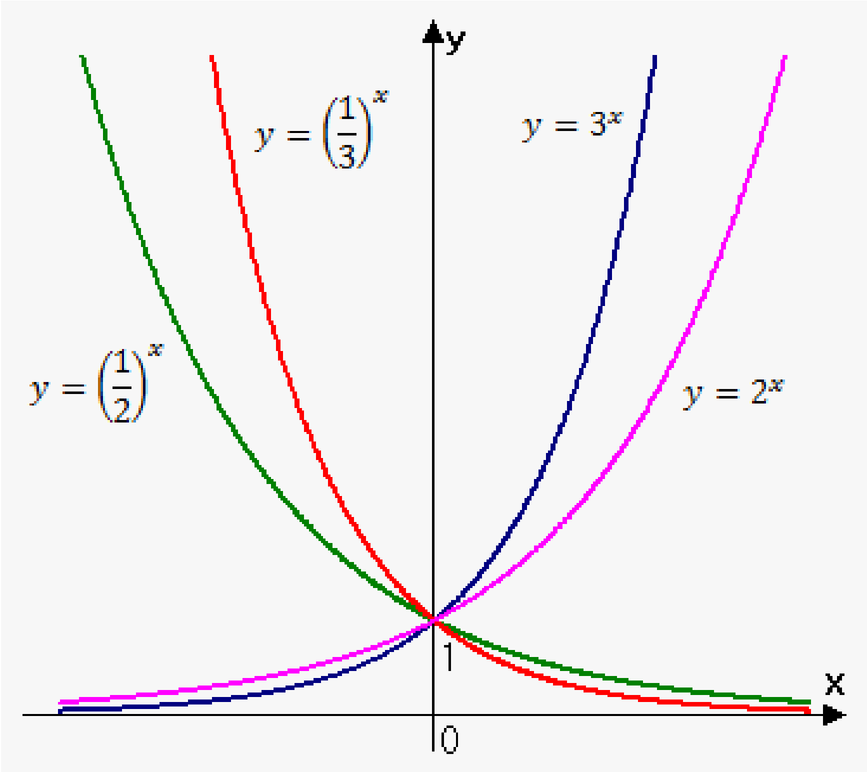
Quotient Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Exponential functions

Definition: The function of the form $f(x) = a^x$, where the *base* a is a positive constant, is called an **exponential function**.

Let's recall what that means.



Laws of Exponents: If a and b are positive numbers and x and y are any real numbers, then

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

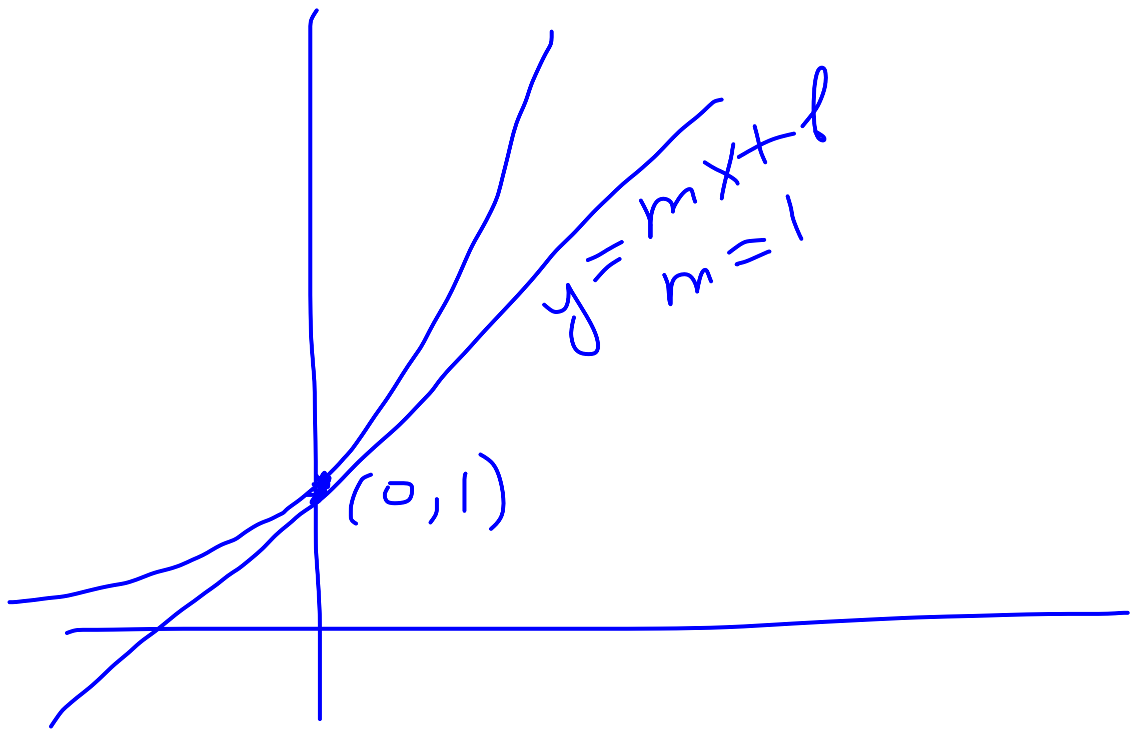
Example: Simplify $\frac{\sqrt{a\sqrt[5]{b}}}{\sqrt[5]{ab}}$

$$= \frac{a^{\frac{1}{2}}\left(b^{\frac{1}{5}}\right)^{\frac{1}{2}}}{a^{\frac{1}{5}}\, b^{\frac{1}{5}}} = \frac{a^{\frac{1}{2}}\, b^{\frac{1}{10}}}{a^{1/5}\, b^{1/5}}$$

$$= a^{\frac{1}{2}-\frac{1}{5}}\, b^{\frac{1}{10}-\frac{1}{5}} = a^{\frac{3}{10}}\, b^{-\frac{1}{10}}$$

The number *e*

$y = e^x$



$$e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\approx 2.71828$$

 How can we get new functions from the ones we know?

Transformations of functions

<u>Vertical and Horizontal Shifts</u>: Suppose $c > 0$. To obtain the graph of

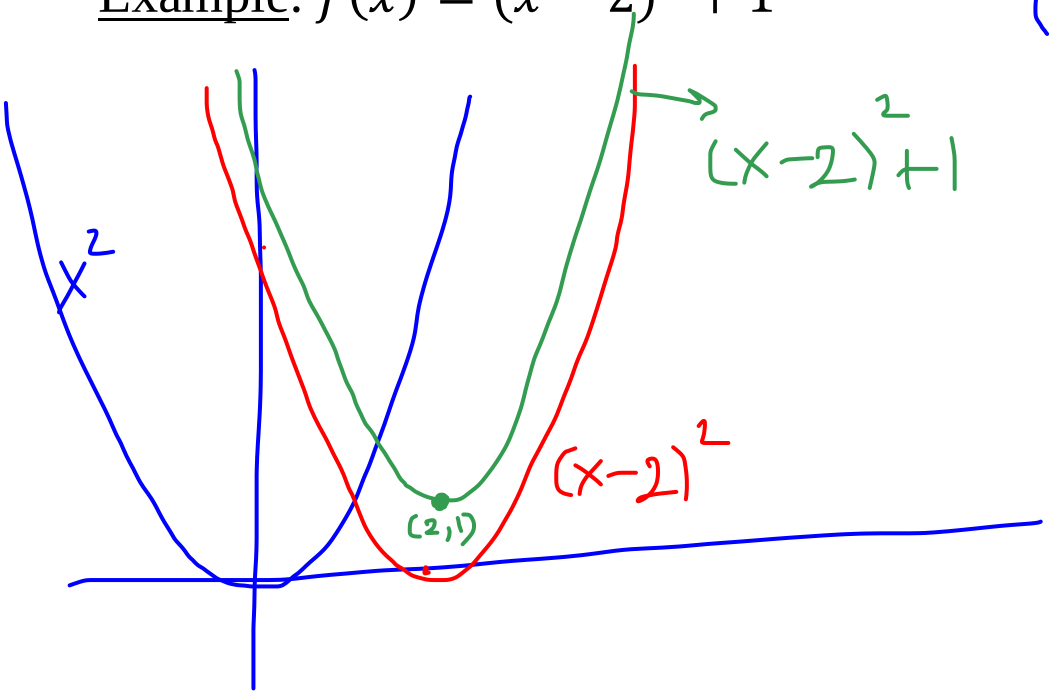
- $y = f(x) \pm c$, shift the graph of $y = f(x)$ a distance c units upward/downward
- $y = f(x \pm c)$, shift the graph of $y = f(x)$ a distance c units to the left/right

<u>Example</u>: $f(x) = (x-2)^2 + 1$

$(2,1)$ - vertex

$(x-k)^2 + h$

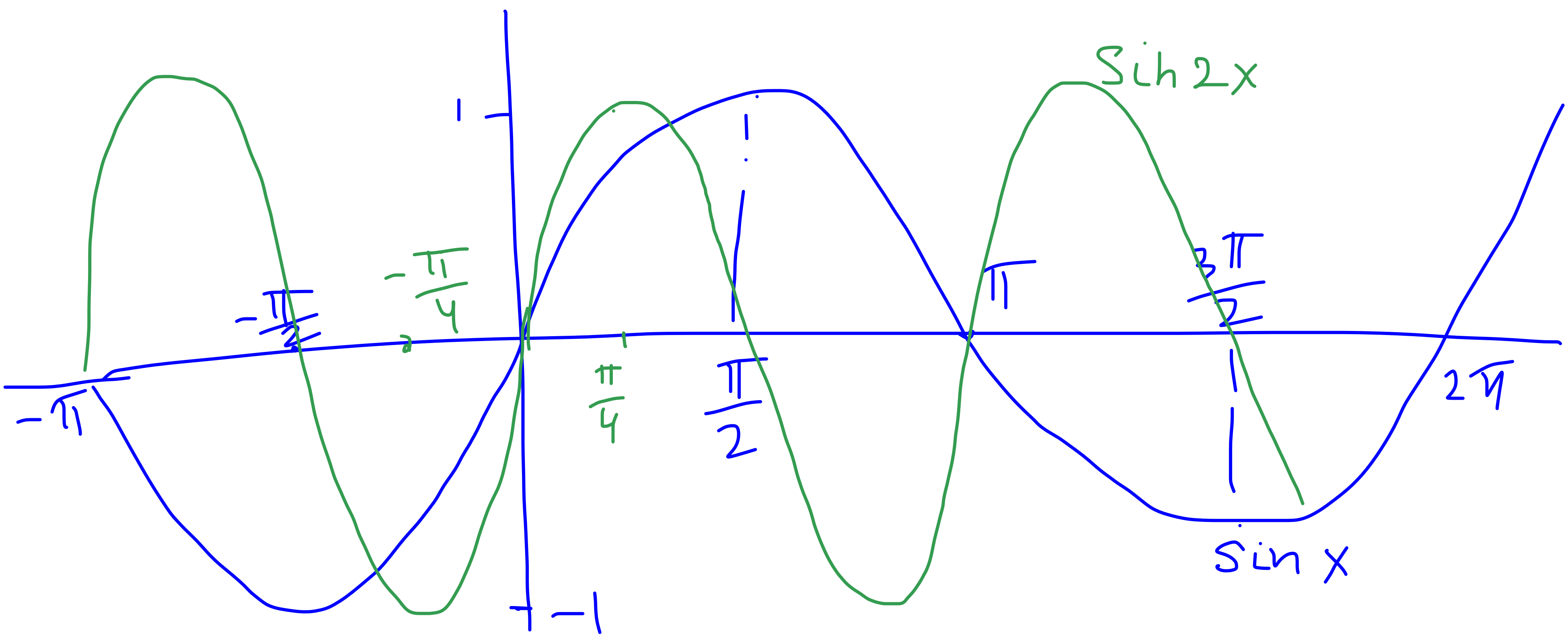
(k,h) - vertex



Vertical and Horizontal Stretching: Suppose $c > 0$. To obtain the graph of

- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
- $y = \frac{1}{c}f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
- $y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
- $y = f\left(\frac{x}{c}\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

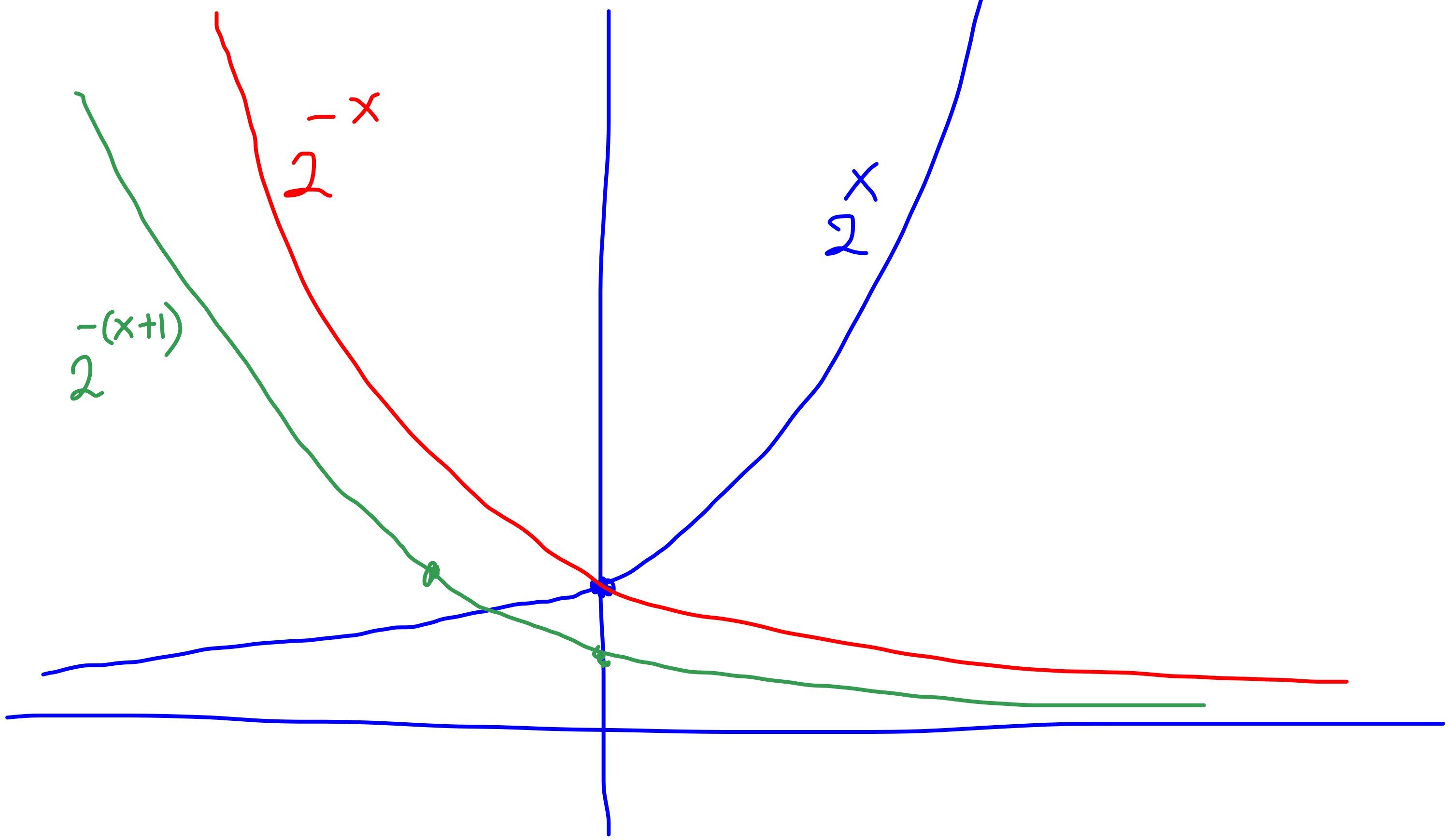
Example: $y = \sin 2x$



Reflecting: To obtain the graph of

- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Example: $y = 2^{-x-1} = 2^{-(x+1)}$



Combinations of functions

$$(f \pm g)(x) = f(x) \pm g(x) \text{ (\textbf{sum/defference})}$$

$$(fg)(x) = f(x)g(x) \text{ (\textbf{product})}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0 \text{ (\textbf{quotient})}$$

$$(f \circ g)(x) = f(g(x)) \text{ (\textbf{composite function})}$$

$f \circ g \neq g \circ f$

<u>Example</u>: If $f(x) = e^x$ and $g(x) = sin^2 x$, find $f \circ g$, $g \circ f$, and $g + f \circ f$.

$f \circ g = f(g(x)) = f(\sin^2 x) = e^{\sin^2 x}$

$\#$

$g \circ f = g(f(x)) = g(e^x) = \sin^2(e^x)$

$g + f \circ f = \sin^2 x + e^{e^x}$

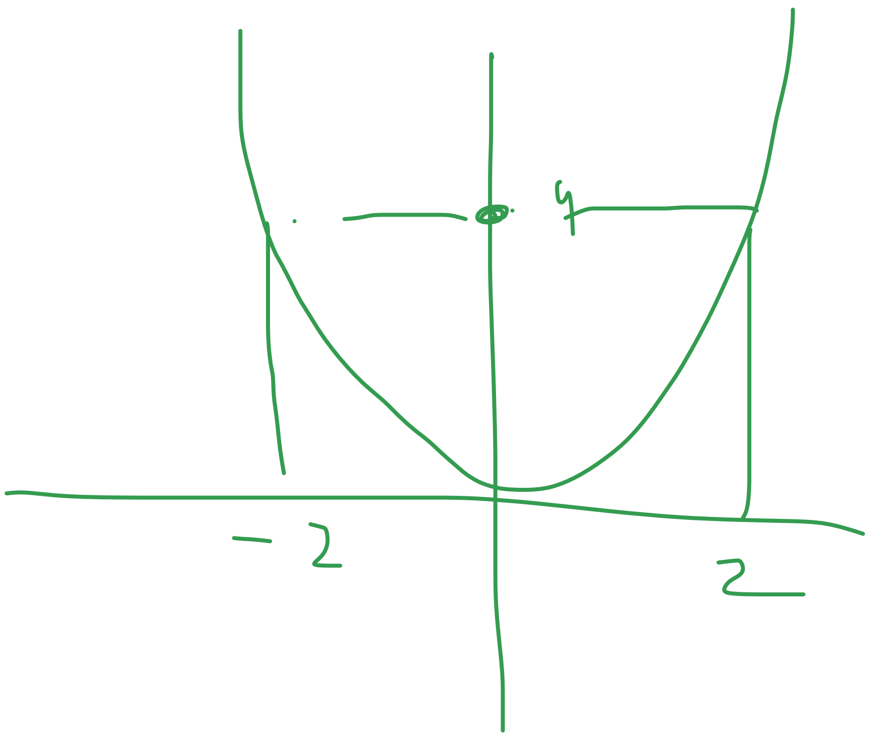
What about $f \circ f \circ f$? $= f(f(f(x))) = e^{e^{e^x}}$

Inverse functions

Definition: A function f is called a one-to-one function if it never takes on the same value twice, i.e.

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

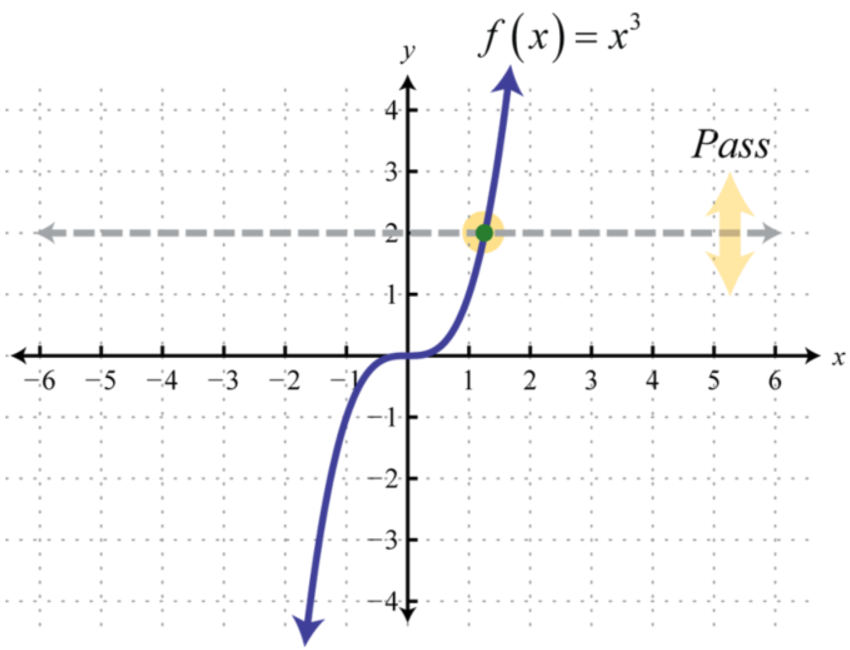
Example: $y = x^2$. Is it one-to-one? NO



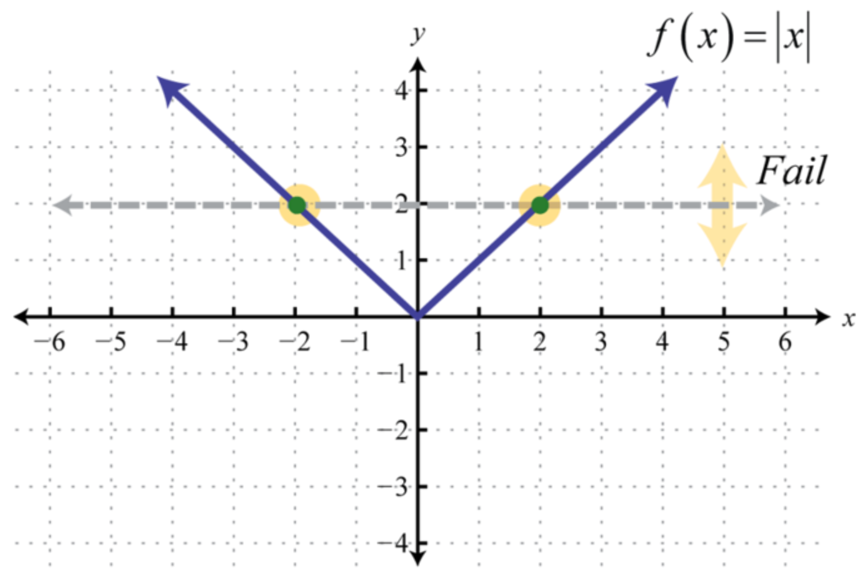
$(-2)^2 = 4 = (2)^2$

 How to check?

<u>Horizontal line test</u>: A function is one-to-one if and only if no horizontal line intersects its graph more than once.



One-to-one Function: Yes



One-to-one Function: No

Definition: Let f be one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y \text{ for any } y \in B$$

Note: $f^{-1}(x) \neq \frac{1}{f(x)}$

Example: Given that $f(x)$ is one-to-one, and $f(0) = -1$, $f(2) = 0$, $f(3) = 2$. Find $f^{-1}(-1)$, $f^{-1}(0)$, and $f(f^{-1}(2))$.

$f^{-1}(-1) = x$

$f(x) = -1 \Rightarrow x = 0$, so $f^{-1}(-1) = 0$

$f^{-1}(0) = 2$

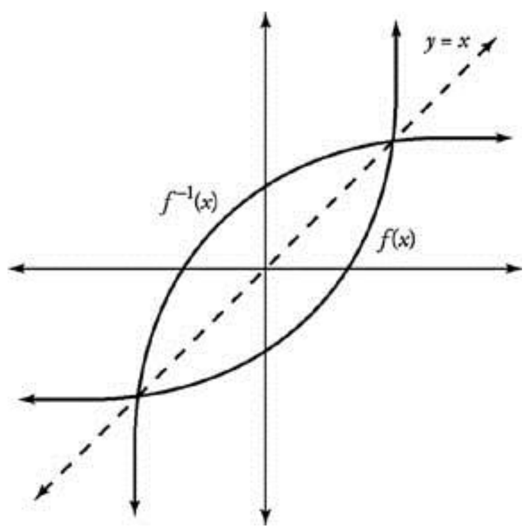
$f(f^{-1}(2)) = f(3) = 2$

Note: Inverse functions have the unique property that, when composed with their original functions, both functions cancel out. Mathematically, this means that

$$f^{-1}\big(f(x)\big) = x, \qquad x \in A$$

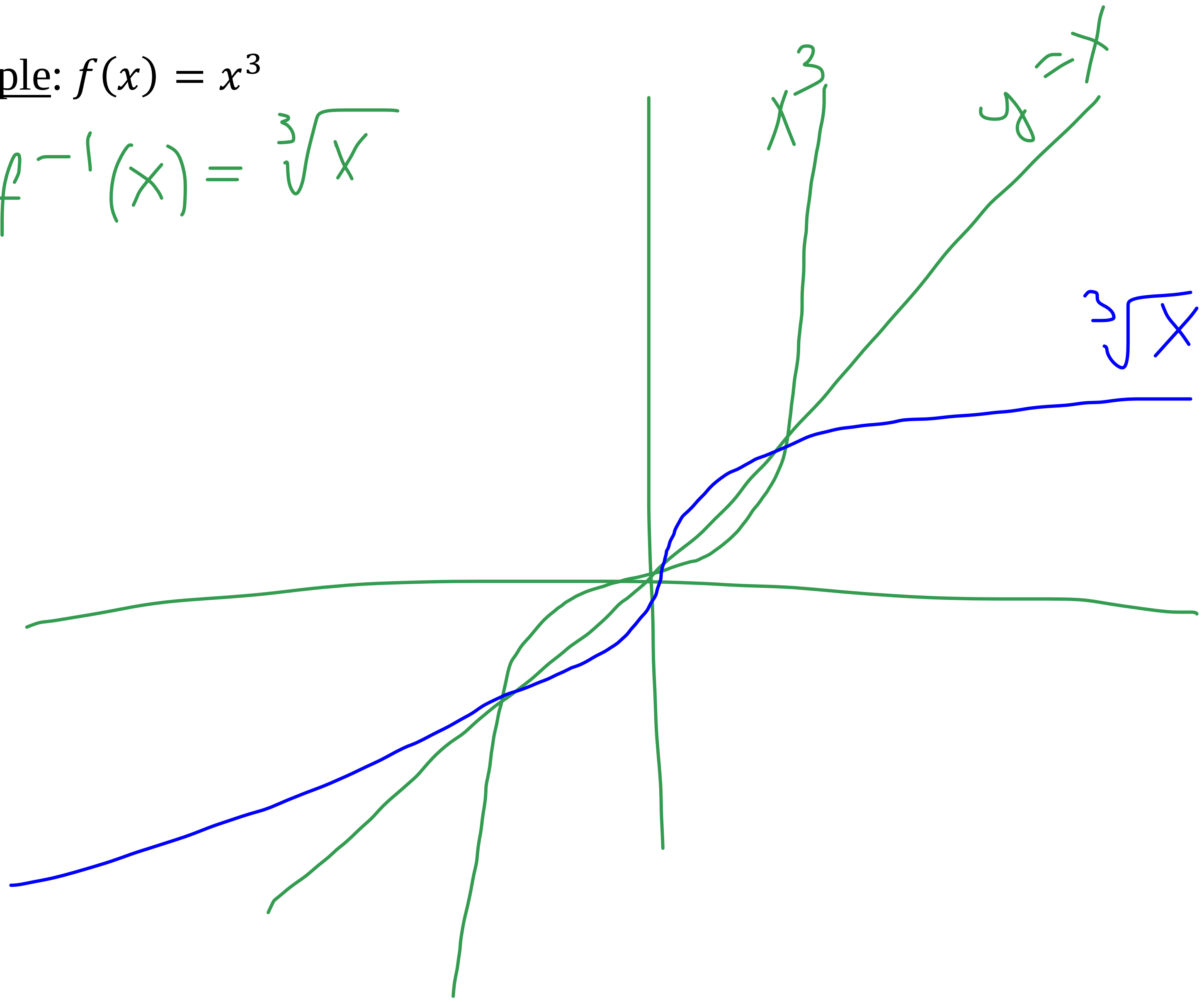
$$f\big(f^{-1}(x)\big) = x, \qquad x \in B$$

Since functions and inverse functions contain the same numbers in their ordered pair, just in reverse order, their graphs will be reflections of one another across the line $y = x$:



Example: $f(x) = x^3$

$f^{-1}(x) = \sqrt[3]{x}$



How to find the inverse function?

To find the inverse function for a one-to-one function, follow these steps:

1. Rewrite the function using y instead of $f(x)$.

2. Solve the equation for x in term of y.

3. Switch the x and y variables

4. The resulting equation is $y = f^{-1}(x)$

5. Make sure that your resulting inverse function is one-to-one. If it isn't, restrict the domain to pass the horizontal line test.

Example: Given $f(x) = \sqrt{x+3}$, find $f^{-1}(x)$.

1. $y = \sqrt{x+3}$

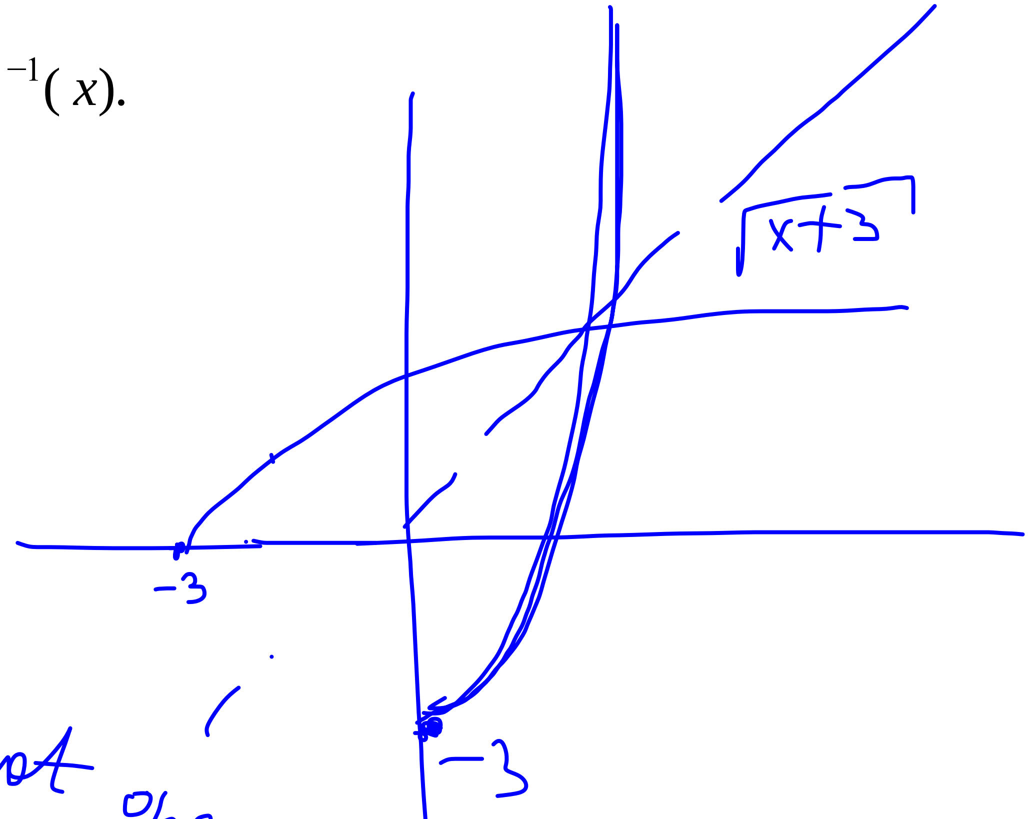
2. $y^2 = x+3$

$x = y^2 - 3$

3. $y = x^2 - 3$

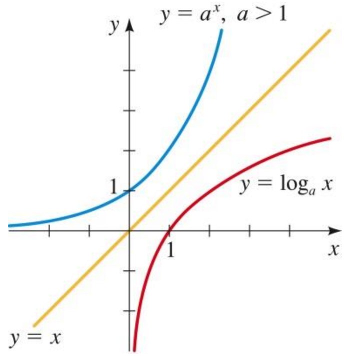
4. $f^{-1}(x) = x^2 - 3 \rightarrow$ not one-to-one

restriction: $x \geq 0$



Note: $x \geq 0$ for $f^{-1}(x)$. Without this restriction, $f^{-1}(x)$ would not pass the horizontal line test. It obviously must be one-to-one, since it must possess an inverse of $f(x)$. You should use that portion of the graph because it is the reflection of $f(x)$ across the line $y = x$, unlike the portion on $x < 0$.

Examples of inverse functions you need to know



- **Logarithmic functions**

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is one-to-one, so it has an inverse function f^{-1} called the **logarithmic function with base *a*.**

Notation: $\log_a$

Thus,

$$f^{-1}(x) = \log_a x = y \iff f(y) = a^y = x$$

Cancellation property:

$$f^{-1}\big(f(x)\big) = \log_a(a^x) = x, \qquad x \in \mathbb{R}$$

$$f\big(f^{-1}(x)\big) = a^{\log_a x} = x, \quad x \in \mathbb{R}$$

Laws of logarithms: Given $x, y \in \mathbb{Z}^+$ (positive integers)

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^r = r \log_a x, r \in \mathbb{R}$

Note: $\log_a a = 1$

Example: Evaluate $\log_2 5 - \log_2 40 - \underbrace{\log_2 1}_{=0}$

$$= \log_2 \frac{5}{40} - 0$$

$$= \log_2 \frac{1}{8} = \log_2 2^{-3} = -3 \underbrace{\log_2 2}_{1}$$

$$= -3$$

Definition: The logarithm with base e is called the **natural logarithm**.

Notation: $\log_e x = \ln x$

So,

$$\ln x = y \Leftrightarrow e^y = x$$

$$\ln e^x = x, \qquad x \in \mathbb{R}$$

$$e^{\ln x} = x, \qquad x > 0$$

$$\ln e = 1$$

Example: Solve $e^{3-x} = 7$

$\ln e^{3-x} = \ln 7$

$3 - x = \ln 7$

$x = 3 - \ln 7$

$e^{\ln(\ln x)} = e^5$

$\ln x = e^5$

$e^{\ln x} = e^{e^5}$

$x = e^{e^5}$

Change of base formula:

$$\log_a x = \frac{\ln x}{\ln a}, \quad a > 0, a \neq 1$$

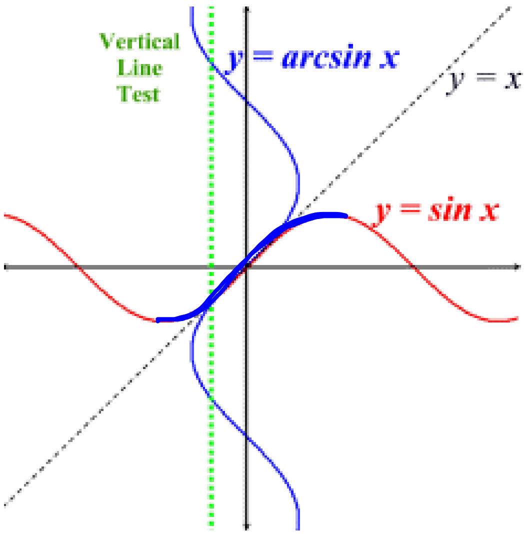
Example: Evaluate $\log_8 3$

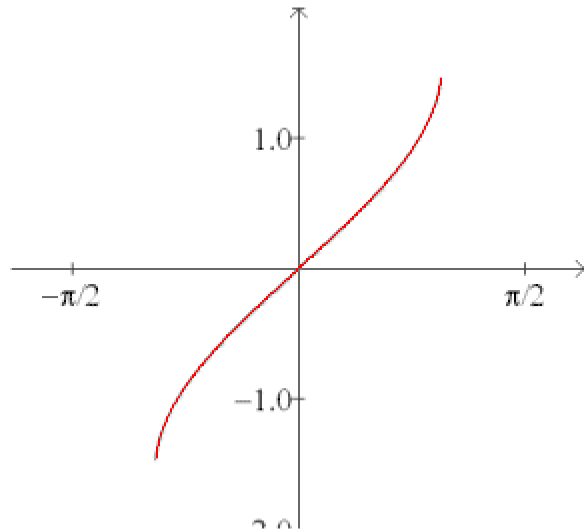
$$= \frac{\ln 3}{\ln 8} = 0.528$$

calculator

- **Inverse trigonometric functions**

Inverse sine function or acrsine function: $\sin^{-1} x$





Domain: [-1,1]
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \sin x$ is not one-to-one, but for $-\pi/2 \leq x \leq \pi/2$ it is.

So we have

$$\sin^{-1} x = y \Leftrightarrow \sin y = x \text{ and } -\pi/2 \le y \le \pi/2$$

$$\sin^{-1}(\sin x) = x, \ -\pi/2 \le x \le \pi/2$$

$$\sin(\sin^{-1} x) = x, \ -1 \le x \le 1$$

<u>Example</u>: Evaluate

(a) $\sin^{-1} 1/2$ $= x$

$\sin(\sin^{-1}\frac{1}{2}) = \sin x$

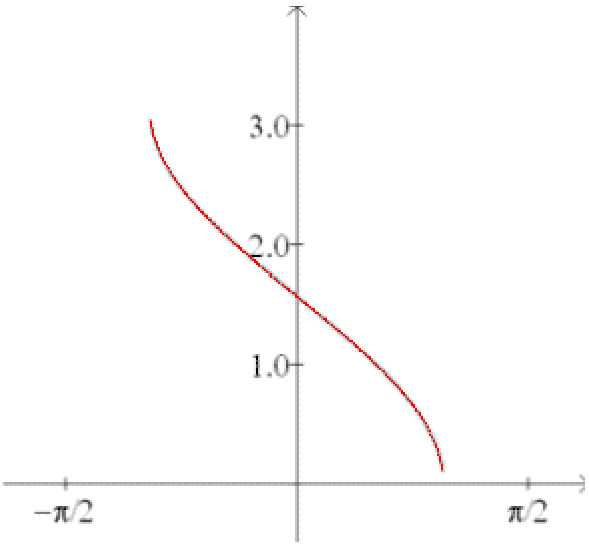
$\frac{1}{2} = \sin x \Rightarrow x = \frac{\pi}{6}$

(b) $\cos \sin^{-1} \frac{1}{\sqrt{2}}$ $= \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\sin^{-1}\frac{1}{\sqrt{2}} = x$

$\frac{1}{\sqrt{2}} = \sin x \Rightarrow x = \frac{\pi}{4}$

Similarly we can define inverse functions for other trigonometric functions:

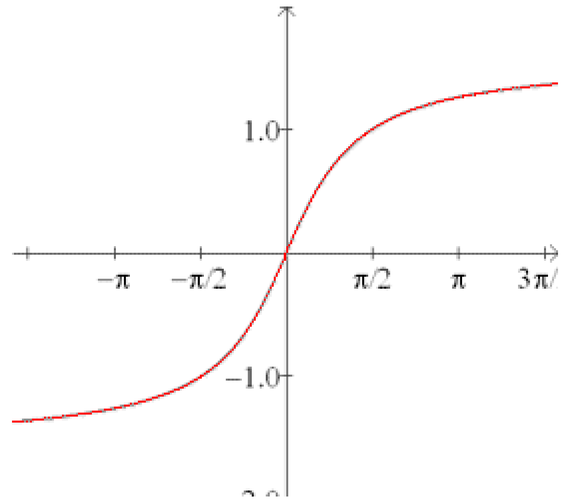


$f(x) = \cos^{-1}(x)$

$f(x) = \arccos(x)$

Domain: [-1,1]

Range: $[0, \pi]$

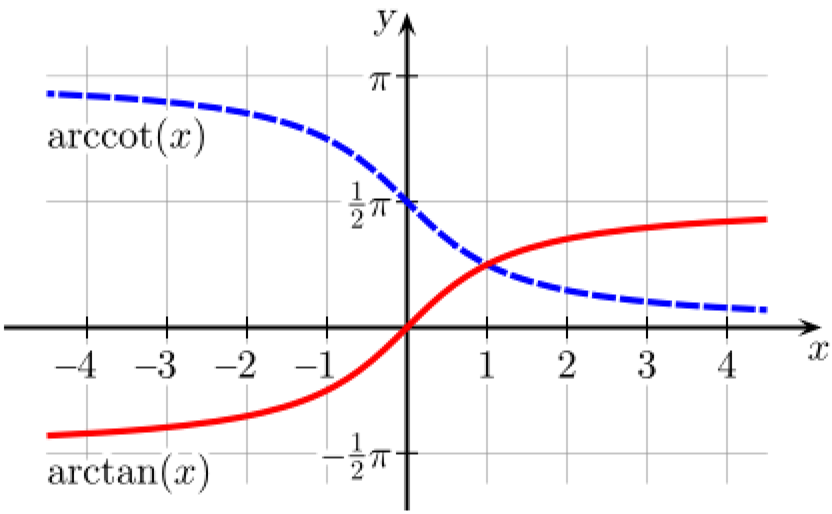


$f(x) = \tan^{-1}(x)$

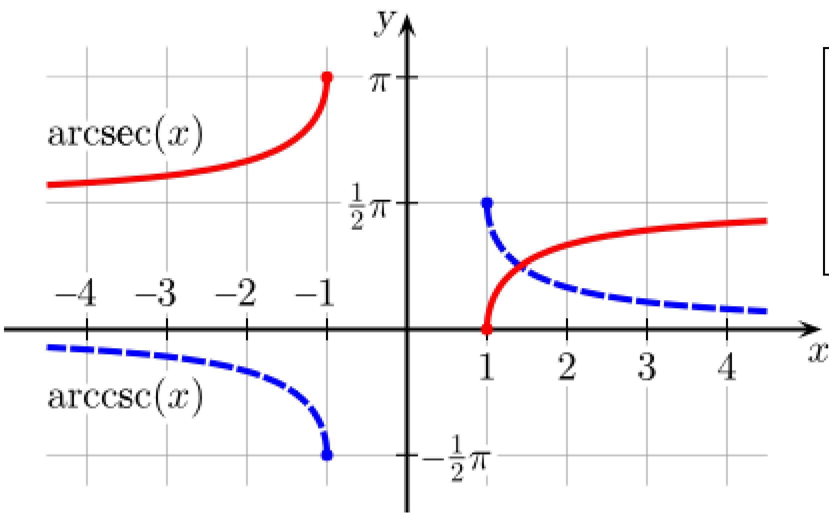
$f(x) = \arctan(x)$

Domain: $(-\infty, \infty)$

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

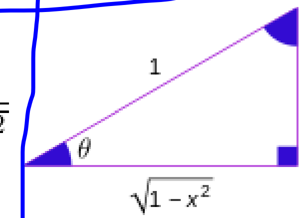
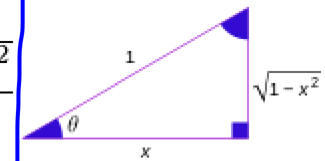
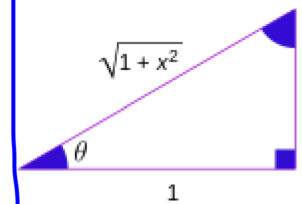
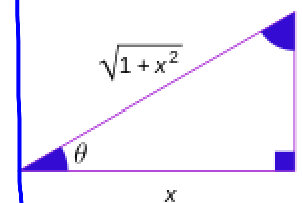
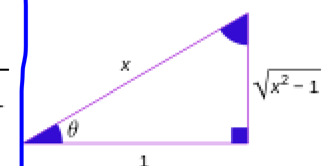
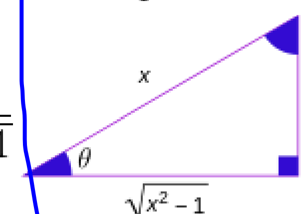


$$y = \cot^{-1} x\,, x \in \mathbb{R} \iff \cot y = x, \qquad y \in (0, \pi)$$



$$y = \sec^{-1} x\,, |x| \geq 1 \quad \iff \quad \sec y = x, y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$y = \csc^{-1} x\,, |x| \geq 1 \quad \iff \csc y = x, y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	Diagram
$\arcsin x$	$\sin(\arcsin x) = x$	$\cos(\arcsin x) = \sqrt{1-x^2}$	$\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$	
$\arccos x$	$\sin(\arccos x) = \sqrt{1-x^2}$	$\cos(\arccos x) = x$	$\tan(\arccos x) = \dfrac{\sqrt{1-x^2}}{x}$	
$\arctan x$	$\sin(\arctan x) = \dfrac{x}{\sqrt{1+x^2}}$	$\cos(\arctan x) = \dfrac{1}{\sqrt{1+x^2}}$	$\tan(\arctan x) = x$	
$\operatorname{arccot} x$	$\sin(\operatorname{arccot} x) = \dfrac{1}{\sqrt{1+x^2}}$	$\cos(\operatorname{arccot} x) = \dfrac{x}{\sqrt{1+x^2}}$	$\tan(\operatorname{arccot} x) = \dfrac{1}{x}$	
$\operatorname{arcsec} x$	$\sin(\operatorname{arcsec} x) = \dfrac{\sqrt{x^2-1}}{x}$	$\cos(\operatorname{arcsec} x) = \dfrac{1}{x}$	$\tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$	
$\operatorname{arccsc} x$	$\sin(\operatorname{arccsc} x) = \dfrac{1}{x}$	$\cos(\operatorname{arccsc} x) = \dfrac{\sqrt{x^2-1}}{x}$	$\tan(\operatorname{arccsc} x) = \dfrac{1}{\sqrt{x^2-1}}$	

General solutions

Note: trigonometric functions are periodic.

This periodicity is reflected in the general inverses:

$$\sin(y) = x \;\Leftrightarrow\; y = \arcsin(x) + 2k\pi \text{ or } y = \pi - \arcsin(x) + 2k\pi,\ k \in \mathbb{Z}$$

or

$$\sin(y) = x \;\Leftrightarrow\; y = (-1)^k \arcsin(x) + k\pi$$

$$\cos(y) = x \;\Leftrightarrow\; y = \arccos(x) + 2k\pi \text{ or } y = 2\pi - \arccos(x) + 2k\pi$$

or

$$\cos(y) = x \;\Leftrightarrow\; y = \pm\arccos(x) + 2k\pi$$

$$\tan(y) = x \;\Leftrightarrow\; y = \arctan(x) + k\pi$$

$$\cot(y) = x \;\Leftrightarrow\; y = \operatorname{arccot}(x) + k\pi$$

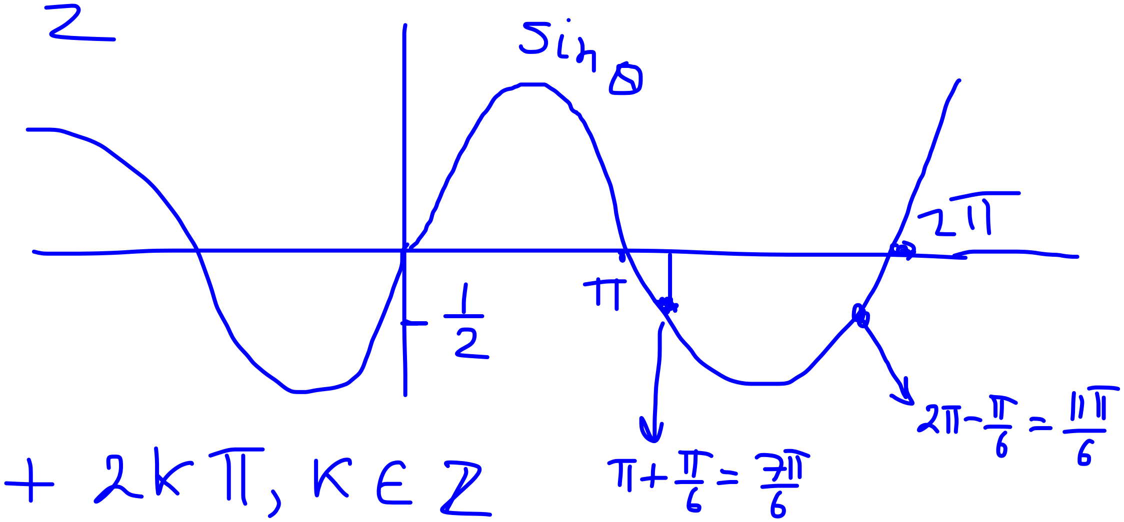
$$\sec(y) = x \;\Leftrightarrow\; y = \operatorname{arcsec}(x) + 2k\pi \text{ or } y = 2\pi - \operatorname{arcsec}(x) + 2k\pi$$

$$\csc(y) = x \;\Leftrightarrow\; y = \operatorname{arccsc}(x) + 2k\pi \text{ or } y = \pi - \operatorname{arccsc}(x) + 2k\pi$$

Example: Solve equation $\quad 2\sin 2x + 1 = 0$

$$\sin 2x = -\frac{1}{2}$$

$$\theta = 2x$$



$$\theta = 2x = \frac{7\pi}{6} + 2k\pi, \ k \in \mathbb{Z}$$

$$\frac{11\pi}{6} + 2k\pi, \ k \in \mathbb{Z}$$

$$x = \frac{7\pi}{12} + k\pi, \ k \in \mathbb{Z}$$

$$\frac{11\pi}{12} + k\pi, \ k \in \mathbb{Z}$$