1. **(14 marks)**

(a) **(5 marks)** Find the domain of the function

$$f(x) = \frac{1}{\ln(x^2 - 1)}$$

$\ln(x^2-1) \neq 0$

$x^2 - 1 \neq 1$

$x^2 \neq 2$

$x \neq \pm\sqrt{2}$

$x^2 - 1 > 0$

$(x-1)(x+1) > 0$

$+ \quad - \quad +$

$-1 \quad +1$

$x \in (-\infty, -1) \cup (1, \infty)$

Domain of $f(x) = (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, -1) \cup (1, \sqrt{2}) \cup (\sqrt{2}, \infty)$

(b) (**4 marks**) Determine whether the given function is even, odd or neither. Justify your answer.

$$f(x) = \frac{x^3 \sin(1 - x^2)}{\log_2(1 + |x|)}$$

$$f(-x) = \frac{(-x)^3 \sin(1 - (-x)^2)}{\log_2(1 + |-x|)} =$$

$$= -\frac{x^3 \sin(1 - x^2)}{\log_2(1 + |x|)} = -f(x)$$

So, $f(x)$ is odd

(c) (**5 marks**) Solve for x.

$$\log_2[x(x-3)] = 2$$

$$2^{\log_2[x(x-3)]} = 2^2$$

$$x(x-3) = 4$$

$$x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

$$x = -1,\ 4 \in \text{domain}$$

Domain: $x(x-3) > 0$

$x \in (-\infty, 0) \cup (3, \infty)$

2. **(15 marks)**

(a) **(5 marks)** Use the **precise definition of limit ($\varepsilon - \delta$ definition)** to prove that

$$\lim_{x\to 10}(3x+5)=35$$

Need to show: for any $\varepsilon > 0$ there is $\delta > 0$ s.t.

if $0 < |x-10| < \delta$, then $|f(x)-35| < \varepsilon$,

where $f(x) = 3x+5$

1. Find δ

$|f(x)-35| < \varepsilon$

$|3x+5-35| < \varepsilon$

$|3x-30| < \varepsilon$

$3|x-10| < \varepsilon$

$|x-10| < \frac{\varepsilon}{3}$

Choose $\delta = \frac{\varepsilon}{3}$

2. Check δ works

If $0 < |x-10| < \delta$

$\Rightarrow$ $3|x-10| < 3\delta$

$|3x-30| < 3\delta$

$|3x+5-35| < 3\delta = 3\cdot\frac{\varepsilon}{3} = \varepsilon$

$|f(x)-35| < \varepsilon$

(b) (**10 marks**) Calculate the following two limits (**5 marks each**):

$$(1) \quad \lim_{x\to 3} \frac{2-\sqrt{1+x}}{x-3} \cdot \frac{2+\sqrt{1+x}}{2+\sqrt{1+x}}$$

$$= \lim_{x\to 3} \frac{4-(1+x)}{(x-3)(2+\sqrt{1+x})}$$

$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{\cancel{(x-3)}(2+\sqrt{1+x})}$$

$$= -\frac{1}{2+\sqrt{1+3}} = -\frac{1}{4}$$

(2) $\lim_{x\to\infty}(x)^{1/x}$

$$y=(x)^{1/x}$$

$$\ln y=\frac{1}{x}\ln x$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln x}{x}\overset{H}{=}\lim_{x\to\infty}\frac{1/x}{1}$$

$$=\lim_{x\to\infty}\frac{1}{x}=0$$

$$\lim_{x\to\infty}(x)^{1/x}=\lim_{x\to\infty}y=\lim_{x\to\infty}e^{\ln y}=e^{\lim_{x\to\infty}\ln y}=e^{0}=1$$

3. (**8 marks**)

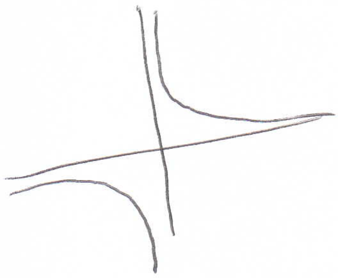
(a) (**3 marks**)

- Give the definition of a function continuous at given point *a* (**2 marks**).
- Give one example of a function that has an infinite discontinuity (**1 mark**).

$f(x)$ is continuous at point a if

$$\lim_{x \to a} f(x) = f(a)$$

Ex. $f(x) = \frac{1}{x}$



$\frac{1}{x}$ has infinite discontinuity at $x=0$

(b) (**5 marks**) Find the value of c that makes function f continuous everywhere.

$$f(x) = \begin{cases} 4 - cx^2, & x \leq -1 \\ 2c - x^3, & x > -1 \end{cases}$$

Show your complete solution.

Need: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (4 - cx^2) = 4 - c$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2c - x^3) = 2c + 1$

$4 - c = 2c + 1$

$3c = 3$

$c = 1$

4. **(15 marks)**

(a) **(5 marks)** Use the **definition of the derivative** to find the derivative of

$$f(x) = \frac{1}{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - (\cancel{x}+h)}{h\,x(x+h)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\,x(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

(b) (**10 marks**) Find y' for the following two functions (**5 marks each**) (you may apply any differentiation rules here):

(1) $y = \ln[\sin(1 - x)]$

$$y' = \frac{1}{\sin(1-x)} \cos(1-x) \ (-1)$$

$$= -\frac{\cos(1-x)}{\sin(1-x)}$$

$$= -\cot(1-x)$$

(2) $y = x^x$

$$(\ln y)' = (x \ln x)'$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + \ln x$$

$$y' = (1 + \ln x) y$$

$$y' = x^x (1 + \ln x)$$

5. (**8 marks**)

(a) (**3 marks**) State the Squeeze Theorem.

If $f(x) \leq g(x) \leq h(x)$

and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

Then $\lim_{x \to a} g(x) = L$

(b) (**5 marks**) Use (a) to find $\lim_{x \to 0} x^2 \cos\frac{1}{x}$

$$-1 \leq \cos\frac{1}{x} \leq 1 \quad \cdot x^2$$

$$-x^2 \leq x^2 \cos\frac{1}{x} \leq x^2$$

$$\lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} x^2$$

By the squeeze Thm,

$$\lim_{x \to 0} x^2 \cos\frac{1}{x} = 0$$

6. (**10 marks**) Given function

$$f(x) = \frac{e^x}{x}$$

(a) (**4 marks**) Find the horizontal and vertical asymptotes (if any).

$$\lim_{x \to \infty} \frac{e^x}{x} \overset{H}{=} \lim_{x \to \infty} \frac{e^x}{1} = \infty$$

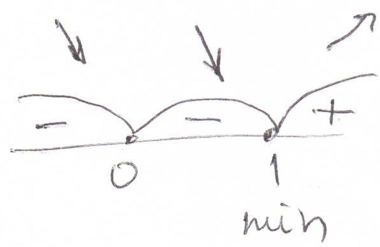
$$\lim_{x \to -\infty} \frac{e^x}{x} = \lim_{x \to -\infty} e^x \cdot \frac{1}{x} = 0 \cdot 0 = 0 \Rightarrow y = 0 \text{ is HA}$$

$$\lim_{x \to 0^+} \frac{e^x}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{e^x}{x} = -\infty \Rightarrow x = 0 \text{ is VA}$$

(b) (**6 marks**) Find the intervals where $f(x)$ is increasing or decreasing. State any local maximum or minimum values.

$$f'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$



$x = 1, 0$ are critical points

$f(x)$ is $\nearrow$ on $(1, \infty)$

$\searrow$ on $(-\infty, 0) \cup (0, 1)$

$(1, e)$ is local min

Bonus question (**5 marks**): Sketch the graph of

$$f(x) = \frac{e^x}{x}$$

(rough sketch is O.K.)

$$f''(x) = \frac{[e^x(x-1) + e^x]x^2 - 2xe^x(x-1)}{x^4}$$

$$= \frac{e^x x^3 - 2x^2e^x + 2xe^x}{x^4} = \frac{e^x(x^2-2x+2)}{x^3}$$

$x^2-2x+2 > 0$

$b^2-4ac = 4-4\cdot2\cdot1 = -4 < 0 \Rightarrow$ no x-intercepts
parabola is concave up

$$f''(x) = \frac{e^x(x^2-2x+2)}{x^3}$$

$0 \notin$ domain

no IP