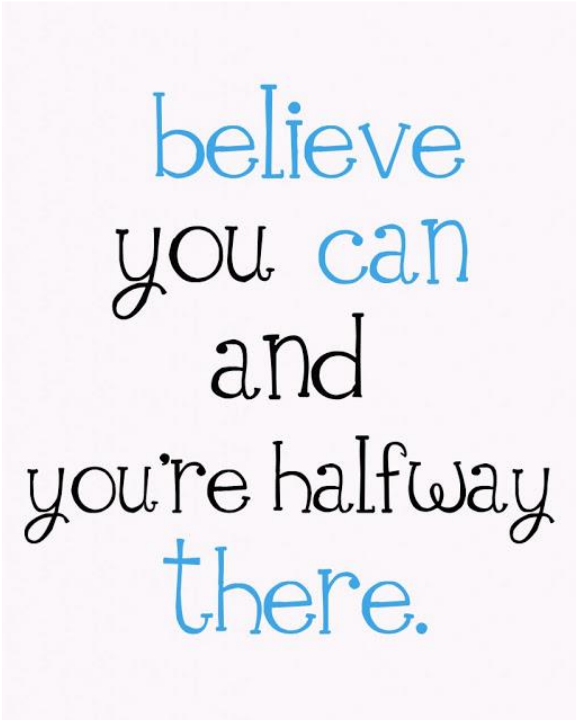
Midterm Review

What to do:

- Read lectures 1-6 (pay attention to the in-class examples) and sections 1.1-1.6, 2.1-2.8, 3.1-3.6, 4.1-4.4 from the textbook
- Do the assigned exercises from the textbook
- Go over the quiz questions
- Use sample test to practice
- Use extra TAs' office hours



Topics to review:

- **Functions and Models**

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- Functions

  - Domain/range
  - Odd/even/neither
  - Increasing/decreasing
  - Transformations
  - Inverse functions
    - Inverse trigonometric
    - Logarithmic

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- Models

  - Linear function
  - Polynomial function
  - Power function
  - Rational function
  - Algebraic function
  - Trigonometric function
  - Exponential function

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Example: Find domain of

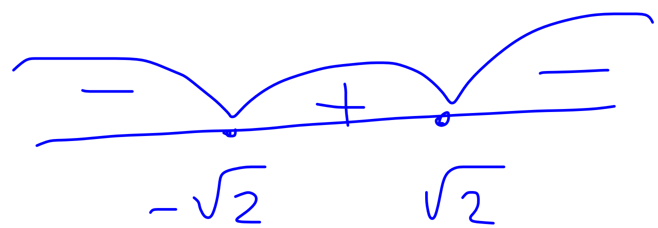
$$f(x) = \frac{\ln\sqrt{2 - x^2}}{x}$$

$x \neq 0$

$\sqrt{2 - x^2} > 0$

$2 - x^2 > 0$

$(\sqrt{2} - x)(\sqrt{2} + x) > 0$



Domain $= (-\sqrt{2}, 0) \cup (0, \sqrt{2})$

Example: Given function

$$f(x) = \cos[x^5 \ln(1 - x^2)]$$

Is it odd, even, or neither?

$$f(-x) = \cos[(-x)^5 \ln(1-(-x)^2)]$$

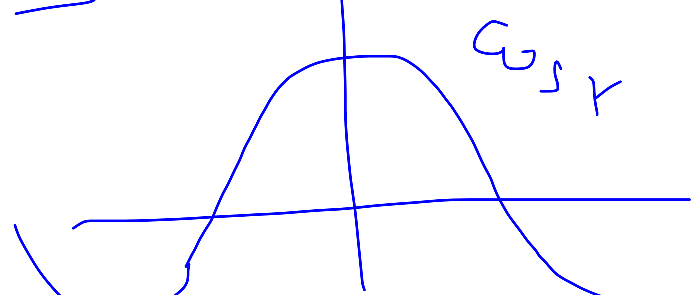
$$= \cos[-x^5 \ln(1-x^2)]$$

$$= \cos[x^5 \ln(1-x^2)] = f(x)$$

$f(x)$ is even

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$



cos x is even

<u>Example</u>: Solve

$$\ln e^{\log_5(x^2-3)} = 1 \quad \ln 1$$

$$\log_5(x^2-3) = 0$$

$$x^2 - 3 = 5^0 = 1$$

$$x^2 - 3 = 1$$

$$x^2 - 4 = 0$$

$$x^2 - 4 = 0$$

$x = \pm 2 \in$ domain

Domain: $x^2 - 3 > 0$

$$(x-\sqrt{3})(x+\sqrt{3}) > 0$$

$$(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$$

- **Limits**
    - One-sided/two-sided limits
    - Limit laws
    - Infinite limits
    - Limits at infinity
    - Vertical/horizontal asymptotes
    - Squeeze Theorem
    - $\varepsilon - \delta$ definition of limit
    - Continuity $\longrightarrow$ definition
        - Types of discontinuities ← removable, jump, infinite } examples
        - Dirichlet example
        - Continuity laws
        - IVT

Example: Calculate

$$\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x^2 + 2x + 4)}$$

$$= \frac{2+2}{2^2 + 2 \cdot 2 + 4} = \frac{4}{12} = \frac{1}{3}$$

$$\frac{0}{0} \qquad \stackrel{H}{=} \lim_{x \to 2} \frac{2x}{3x^2} = \frac{2 \cdot 2}{3 \cdot 2^2} = \frac{1}{3}$$

$$\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right) \cdot \frac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}$$

$$\infty-\infty$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+4x+1-\cancel{x^2}}{\sqrt{x^2+4x+1}+x} =$$

$$= \lim_{x\to\infty} \frac{4x+1}{\sqrt{x^2+4x+1}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{4+\frac{1}{x}}{\sqrt{1+\underbrace{\frac{4}{x}}_{\to 0}+\underbrace{\frac{1}{x^2}}_{\to 0}}+1} = \frac{4}{\sqrt{1}+1} = 2$$

$$\frac{1}{x} \to 0$$

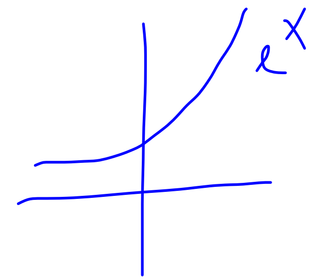
<u>Example</u>: Apply the Squeeze theorem to find

$$\lim_{x\to 0} x^2 e^{\sin\frac{1}{x}} = 0$$

$$f(x) \le g(x) \le h(x)$$

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

$$\Rightarrow \lim_{x\to a} g(x) = L$$



increasing function

$$-1 \le \sin\frac{1}{x} \le 1$$

$$e^{-1} \le e^{\sin\frac{1}{x}} \le e^{1} \quad \cdot x^2$$

$$x^2 e^{-1} \le x^2 e^{\sin\frac{1}{x}} \le x^2 e$$

$x \to 0$: $x^2 e^{-1} \to 0$, $x^2 e \to 0$

$x^2 e^{\sin\frac{1}{x}} \xrightarrow{x\to 0} 0$ by the squeeze Thm

Example: Use $\varepsilon - \delta$ definition of a limit to prove that

$$\lim_{x \to 2}(14 - 5x) = 4$$

$\forall$ $\exists$

For any $\varepsilon > 0$ there is $\delta > 0$ s.t.

if $0 < |x-2| < \delta$ then $|f(x) - 4| < \varepsilon$

Find δ s.t.

$|f(x) - 4| < \varepsilon$

$|14 - 5x - 4| < \varepsilon$

$|10 - 5x| < \varepsilon$

$|-5||x-2| < \varepsilon$

$|x-2| < \frac{\varepsilon}{5}$

Choose $\delta = \frac{\varepsilon}{5}$

whenever $|x-2| < \delta$

$5|x-2| < 5\delta$

$|14 - 5x - 4| < 5\delta$

$|f(x) - 4| < 5 \cdot \frac{\varepsilon}{5}$

$|f(x) - 4| < \varepsilon$

Example: Use IVT to show that there is a root of the equation

$$\cos\sqrt{x} = e^{x} - 2$$

on interval (0, 1).

IVT: $f(a) \le N \le f(b)$, then there is $c \in (a, b)$ s.t. $f(c) = N$

$f(x) = \cos\sqrt{x} - e^{x} + 2$

$f(0) = 1 - e^{0} + 2 = 2$

$f(1) = \cos 1 - e^{1} + 2 = -0.2$

$-0.2 < 0 < 2$

N

By IVT, there is $c \in (0, 1)$ s.t. $f(c) = 0$

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- **Derivatives**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - Definition of derivative
  - Functions differentiable/ not differentiable
  - Higher derivatives
  - Differentiation rules
    - Power rule
    - Product rule
    - Quotient rule
    - Chain rule
  - Derivatives of exponential functions
  - Derivatives of trigonometric functions
  - Implicit differentiation
  - Derivatives of inverse trigonometric functions
  - Derivatives of logarithmic functions
  - Logarithmic differentiation

<u>Example</u>: Use the definition of the derivative to find the derivative of

$$f(x) = x^2 - x \qquad f'(x) = 2x - 1$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x} - h - \cancel{x^2} + \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{+2xh + h^2 - h}{h} = \lim_{h \to 0} (2x + h - 1)$$

$$= 2x - 1$$

Example: Find y'.

$$y = \tan\sqrt{1-\ln(x^2-3)}$$

$$y' = \sec^2\sqrt{1-\ln(x^2-3)} \cdot \left[\left(1-\ln(x^2-3)\right)^{1/2}\right]'$$

$$\frac{1}{2}\left(1-\ln(x^2-3)\right)^{-1/2} \cdot \left(1-\ln(x^2-3)\right)'$$

$$-\frac{1}{x^2-3} \cdot 2x$$

$$= \sec^2\sqrt{1-\ln(x^2-3)} \cdot \frac{1}{2}\left(1-\ln(x^2-3)\right)^{-1/2} \cdot \frac{(-2x)}{x^2-3}$$

$$y = (\sin x)^{\ln x}$$

$$(\ln y)' = (\ln x \; \ln \sin x)'$$

$$\frac{1}{y} y' = \ln x \cdot \frac{1}{\sin x} \cos x + \frac{1}{x} \ln \sin x$$

$$y' = \left[\ln x \cdot \cot x + \frac{\ln \sin x}{x} \right] (\sin x)^{\ln x}$$

- **Applications of Differentiation**
    - Minimum/maximum values
    - Fermat's theorem
    - Rolle's theorem
    - MVT
    - Increasing/decreasing test
    - First derivative test
    - Concavities/inflection points
    - Second derivative test
    - L'Hospital's rule
    - Sketching the graph of a function

Example: Calculate

$$\lim_{x\to 0^+} (\cos x)^{1/x^2}$$

1^∞

$y = (\cos x)^{1/x^2}$

$\ln y = \frac{\ln \cos x}{x^2}$

$$\lim \ln y = \lim_{x\to 0^+} \frac{\ln \cos x}{x^2} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{\cos x}\cdot(-\sin x)}{2x}$$

$\frac{0}{0}$

$$= \lim_{x\to 0^+} \frac{-\tan x}{2x} \overset{H}{=} \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

$\frac{0}{0}$

$$\lim_{x\to 0^+} (\cos x)^{1/x^2} = \lim y = \lim e^{\ln y} = e^{\lim \ln y} = e^{-1/2}$$

Example: Sketch $f(x) = \frac{e^x}{x^2}$ (see lecture 6)