Final Review

What to do:

- Read lectures 1-11, and corresponding sections from the textbook
- Go over the midterm review

  midterm test
- Do the assigned exercises from the textbook
- Go over the quiz questions
- Use sample exam to practice
- Use extra TAs' office hours

IVT

Rolle's

$\boxed{\text{MVT}}$

Topics to review:

- **Integrals**
  - Riemann sums
  - Definite integrals
  - FTC, Part I and II
  - Indefinite integrals

$+C$

$$g(x) = \int_a^x f(t)\,dt$$

$$g'(x) = f(x)$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example: Calculate the integral as a limit of Riemann sum. Then check your answer by applying the FTC, Part II.

$$\int_{-1}^{0}(x^2-1)dx$$

$$= F(0)-F(-1) = \left[\frac{x^3}{3}-x\right]_{-1}^{0} = \frac{1}{3}-1 = -\frac{2}{3}$$

FTC, II

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{i^2}{n^2}-\frac{2i}{n}\right]\cdot\frac{1}{n}$$

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

$$= -1 + i\cdot\frac{1}{n} = \frac{i}{n}-1$$

$$= \frac{0-(-1)}{n} = \frac{1}{n}$$

$$f(x_i) = x_i^2 - 1 = \left(\frac{i}{n}-1\right)^2 - 1 = \left(\frac{i}{n}\right)^2 - \frac{2i}{n} + \cancel{1} - \cancel{1}$$

$$= \frac{i^2}{n^2} - \frac{2i}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{i^2}{n^2}-\frac{2i}{n}\right]\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n^2}\sum i^2-\frac{2}{n}\sum i\right]$$

$$\sum i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum i = \frac{n(n+1)}{2}$$

$$= \lim \cancel{\frac{1}{n}}\left[\frac{1}{n^2}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6}-\frac{\cancel{2}}{n}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{1}{3}-1 = -\frac{2}{3}$$

Example: Find the derivative of

$$h(x) = \int_{-e^x}^{x^4} \sin t \, dt$$

FTC, I: $g(x) = \int_a^x f(t)\,dt \Rightarrow g'(x) = f(x)$

$$g(x) = \int_a^{G(x)} f(t)\,dt \Rightarrow g'(x) = f(G(x)) \cdot G'(x)$$

$$h(x) = \int_{-e^x}^{0} \sin t\,dt + \int_0^{x^4} \sin t\,dt$$

$$= -\int_0^{-e^x} \sin t\,dt + \int_0^{x^4} \sin t\,dt$$

$$h'(x) = -\sin(-e^x) \cdot (-e^x) + \sin(x^4) \cdot 4x^3$$

$$= -e^x \sin(e^x) + 4x^3 \sin(x^4)$$

- **Techniques of Integration**
  - Substitution rule
  - Integration by parts
  - Trigonometric integrals
  - Trigonometric substitution
  - Integration of rational functions by partial fractions
  - Improper integrals

<u>Example</u>: Find

$$\int x^3 e^{x^2}\,dx = \int x^2 e^{x^2} \underbrace{x\,dx}_{\frac{1}{2}du}$$

$$u = x^2$$

$$du = 2x\,dx$$

$$= \frac{1}{2}\int u e^u\,du = \frac{1}{2}\left[u e^u - \int e^u\,du \right]$$

$$w = u \qquad dv = e^u\,du$$

$$dw = du \qquad v = e^u$$

$$= \frac{1}{2} u e^u - e^u + C$$

$$= \frac{1}{2} x^2 e^{x^2} + e^{x^2} + C$$

<u>Example</u>: Calculate

$$\int x^5\sqrt{1+x^3}\,dx = \int x^3\sqrt{1+x^3}\cdot \underbrace{x^2\,dx}_{\frac{1}{3}du}$$

$$u = 1+x^3 \Rightarrow x^3 = u-1$$

$$du = 3x^2\,\cancel{dx}$$

$$= \frac{1}{3}\int (u-1)\sqrt{u}\,du$$

$$= \frac{1}{3}\int \left[u^{3/2} - u^{1/2}\right] du$$

$$= \frac{1}{3}\cdot\frac{2}{5}u^{5/2} - \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{15}(1+x^3)^{5/2} - \frac{2}{9}(1+x^3)^{3/2} + C$$

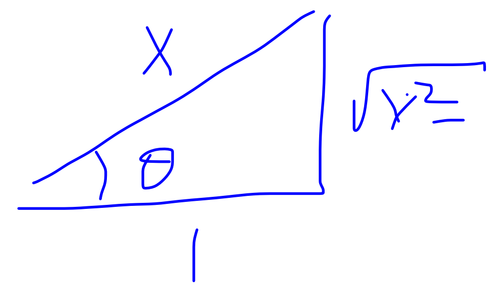
Example: Find

$$\int \frac{x \ln x}{\sqrt{x^2-1}}\,dx$$

$u = \ln x \qquad dv = \dfrac{x}{\sqrt{x^2-1}}\,dx$

$du = \dfrac{1}{x}\,dx \qquad v = \sqrt{x^2-1}$

$$= \sqrt{x^2-1}\,\ln x - \int \frac{\sqrt{x^2-1}}{x}\,dx$$



$x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$\tan\theta = \sqrt{x^2-1}$

$$= \sqrt{x^2-1}\,\ln x - \int \frac{\tan\theta}{\cancel{\sec\theta}}\,\cancel{\sec\theta}\tan\theta\,d\theta$$

$$= \sqrt{x^2-1}\,\ln x - \int \tan^2\theta\,d\theta$$

$[\tan^2\theta = \sec^2\theta - 1]$

$$= \sqrt{x^2-1}\,\ln x - \int [\sec^2\theta - 1]\, d\theta$$

$$= \sqrt{x^2-1}\,\ln x - \int \sec^2\theta\, d\theta + \int d\theta$$

$$= \sqrt{x^2-1}\,\ln x - \tan\theta + \theta + C$$

$$= \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \sec^{-1}x + C$$

Example: Find

$$\int \frac{dx}{\sqrt{x}(2+\sqrt{x})^4}$$

$u = 2+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= 2\int \frac{du}{u^4} = 2\,\frac{u^{-3}}{-3} + C$

$= -\frac{2}{3}\,\frac{1}{(2+\sqrt{x})^3} + C$

Example: Calculate

$$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$$

$\cos^4 x = (1 - \sin^2 x)^2$

$u = \sin^2 x$

$du = 2 \sin x \cos x \, dx$

$$= \frac{1}{2} \int \frac{du}{u^2 + (1-u)^2} = \frac{1}{2} \int \frac{du}{2u^2 - 2u + 1}$$

$$= \int \frac{du}{4u^2 - 4u + 2} = \int \frac{du}{(2u-1)^2 + 1}$$

$w = 2u - 1$

$dw = 2\, du$

$$= \frac{1}{2} \int \frac{dw}{w^2 + 1} = \frac{1}{2} \tan^{-1} w + C = \frac{1}{2} \tan^{-1}(2\sin^2 x - 1) + C$$

Example: Evaluate

$$\int_1^\infty \frac{\ln x}{x^4}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^4}\,dx$$

$u = \ln x \qquad dv = \frac{1}{x^4}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{3}\frac{1}{x^3}$

$$= \lim_{t\to\infty}\left[-\frac{1}{3}\frac{1}{x^3}\ln x \Big|_1^t + \frac{1}{3}\int_1^t \frac{1}{x^4}\,dx \right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{3}\frac{\ln t}{t^3} + \frac{1}{3}\cdot\frac{-1}{3}\frac{1}{x^3}\Big|_1^t \right]$$

$$= \lim_{t\to\infty}\left[\underbrace{-\frac{1}{3}\frac{\ln t}{t^3}}_{} \underbrace{-\frac{1}{9}\cdot\frac{1}{t^3}}_{\to 0} + \frac{1}{9}\right] = \frac{1}{9}$$

$$\lim_{t\to\infty}\frac{\ln t}{t^3} \overset{H}{=} \lim_{t\to\infty}\frac{1/t}{3t^2} = \lim_{t\to\infty}\frac{1}{3t^3} = 0$$

Example: Evaluate

$$\int_{-1}^{1} \frac{dx}{x^2 - 2x}$$

$$\frac{1}{x^2-2x} = \frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$1 = A(x-2) + Bx$$

$$0 = A + B \implies B = \frac{1}{2}$$

$$1 = -2A \implies A = -\frac{1}{2}$$

$$\frac{1}{x^2-2x} = -\frac{1}{2x} + \frac{1}{2(x-2)}$$

$$\int_{-1}^{1} \frac{dx}{x^2-2x} = \int_{-1}^{0} \frac{dx}{x^2-2x} + \int_{0}^{1} \frac{dx}{x^2-2x}$$

$$= \lim_{t \to 0^-} \int_{-1}^{t} + \lim_{t \to 0^+} \int_{t}^{1}$$

$$\lim_{t\to 0^+}\int_t^1 \frac{dx}{x^2-2x} = \lim_{t\to 0^+}\int_t^1 \left[-\frac{1}{2x}+\frac{1}{2(x-2)}\right]dx$$

$$= \lim_{t\to 0^+}\left[-\frac{1}{2}\ln x + \frac{1}{2}\ln|x-2|\right]_t^1$$

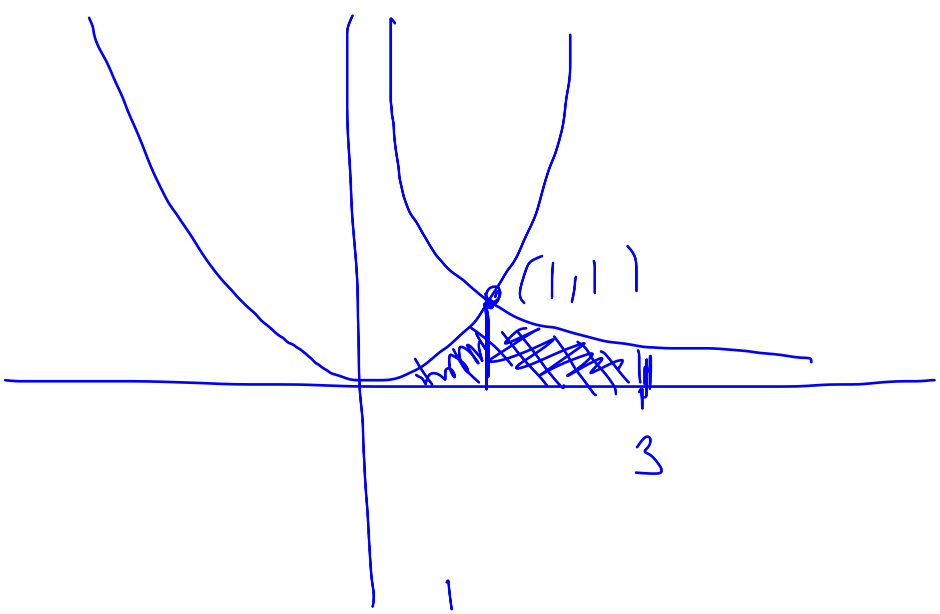
$$= \lim_{t\to 0^+}\left[\frac{1}{2}\ln 2 + \underbrace{\frac{1}{2}\ln t}_{\to -\infty} - \underbrace{\frac{1}{2}\ln|t-2|}_{\to \ln 2}\right]$$

$$= -\infty$$

$\Rightarrow$ divergent

- **Applications of Integration**
  - Areas between curves
  - Volumes

<u>Example</u>: Find the area of the region bounded by $y = \frac{1}{x}, y = x^2, y = 0, x = 3$.

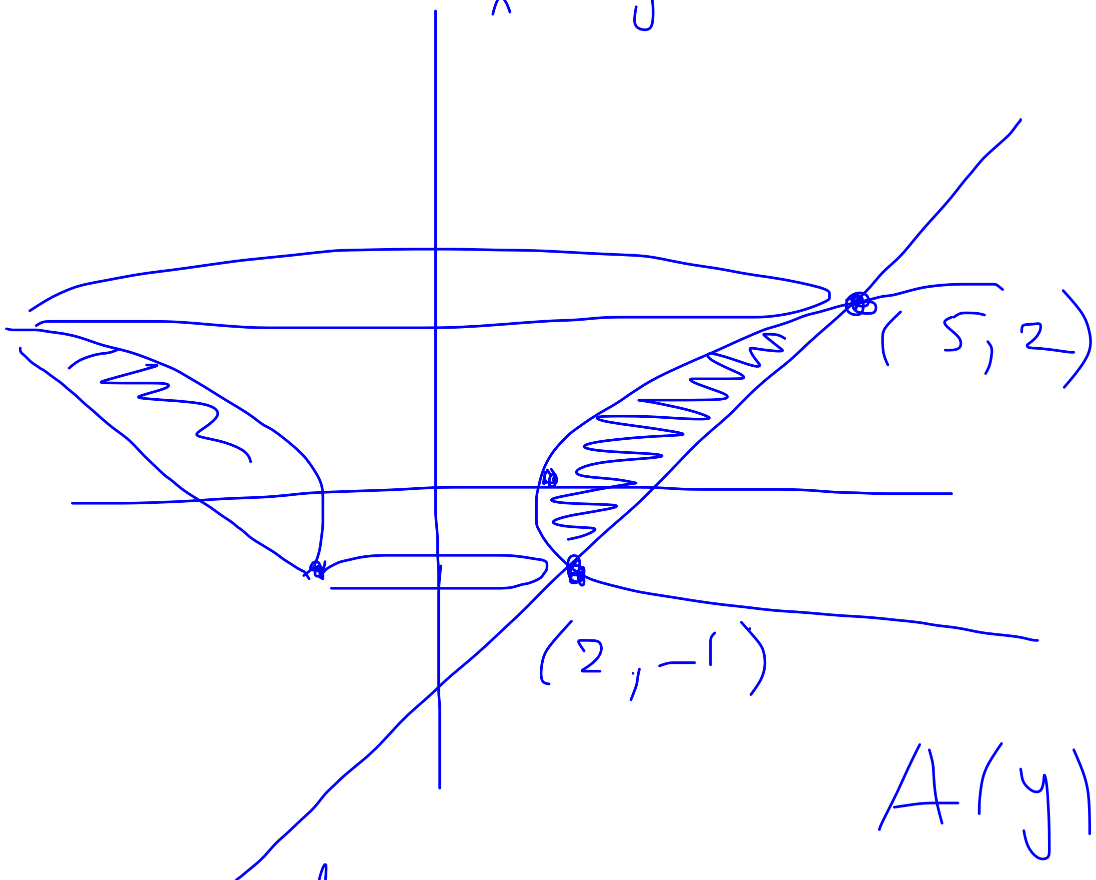


$$x^2 = \frac{1}{x}$$
$$x^3 = 1$$
$$x = 1$$

$$A = \int_0^1 (x^2 - 0)\,dx + \int_1^3 \left(\frac{1}{x} - 0\right) dx$$
$$= \frac{x^3}{3}\Big|_0^1 + \ln x\Big|_1^3 = \frac{1}{3} + \ln 3$$

<u>Example</u>: Find the volume of the solid obtained by rotating the region bounded by $x = 1 + y^2$, $y = x - 3$ about the y-axis.

$x = y + 3$



$1 + y^2 = y + 3$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = 2, -1$

$A(y) = \pi(R^2 - r^2)$

$= \pi\left((y+3)^2 - (1+y^2)^2\right)$

$V = \int_a^b A(y)\,dy = \pi \int_{-1}^{2} \left[(y+3)^2 - (1+y^2)^2\right] dy$

$= \frac{117}{5}\pi$


THE END