Meta-Bayesian Analysis
A Bayesian decision-theoretic analysis of Bayesian inference under model misspecification

Jun Yang
joint work with Daniel Roy

Department of Statistical Sciences
University of Toronto

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Motivation

“All models are wrong, some are useful.”
— George Box

“truth [...] is much too complicated to allow anything but approximations.”
— John von Neumann

➢ Subjectivism Bayesian:
  alluring but impossible to practice when model is wrong

➢ Prior probability = degree of Belief... in what?
  What is a prior?

➢ Is there any role for (subjective) Bayesianism?

Our proposal: More inclusive and pragmatic definition for “prior”.
Our approach: Bayesian decision theory
Example: Grossly Misspecified Model

Setting: Machine learning

data are collection of documents: 

- Model: Latent Dirichlet Allocation (LDA) aka “topic modeling”
- Prior belief: \( \tilde{\pi} \equiv 0 \), i.e., no setting of LDA is faithful to our true beliefs about data.
- Conjugate priors \( \pi(d\theta) \sim \text{Dirichlet}(\alpha) \)

What is the meaning of a prior on LDA parameters?

Pragmatic question: If we use an LDA model (for whatever reason), how should we choose our “prior”? 
Example: Accurate but still Misspecified Model

Setting: Careful Science

data are experimental measurements:

- Model: \((Q_\theta)_{\theta \in \Theta}\), painstakingly produced after years of effort
- Prior belief: \(\tilde{\pi} \equiv 0\),
  i.e., no \(Q_\theta\) is 100% faithful to our true beliefs about data.

What is the meaning of a prior in a misspecified model?
(All models are misspecified.)

Pragmatic question: How should we choose a “prior”?
Standard Bayesian Analysis for Prediction

\( Q_\theta(\cdot) \) Model on \( \mathcal{X} \times \mathcal{Y} \) given parameter \( \theta \)
\( \mathcal{X} \): what you will observe
\( \mathcal{Y} \): what you will then predict

\( \pi(\cdot) \) prior on \( \theta \)

\[(\pi Q)(\cdot) = \int Q_\theta(\cdot) \pi(d\theta)\] Marginal distribution on \( \mathcal{X} \times \mathcal{Y} \)

Believe \( (X, Y) \sim \pi Q \)

The Task

1. Observe \( X \).
2. Choose action \( \hat{Y} \).
3. Suffer loss \( L(\hat{Y}, Y) \)

The Goal

Minimize expected loss

Bayes optimal action minimizes expected loss under the conditional distribution of \( Y \) given \( X = x \), written \( \pi Q(dy|x) \):

\[
\text{BayesOptAction}(\pi Q, x) = \arg \min_a \int L(a, y) \pi Q(dy|x). 
\]

- Quadratic loss \( \rightarrow \) posterior mean.
- Self-information loss (log loss) \( \rightarrow \) posterior \( \pi Q(\cdot|x) \).
Meta-Bayesian Analysis

- \((Q_\theta)_{\theta \in \Theta}\): the model, i.e., a family of distributions on \(\mathcal{X} \times \mathcal{Y}\).
- Don’t believe \(Q_\theta\), i.e., model is misspecified.
- \(P\): represents our true belief on \(\mathcal{X} \times \mathcal{Y}\).

Believe \((X, Y) \sim P\)
But We Will Use \(Q_\theta\) to predict

The Task
1. Choose (surrogate) prior \(\pi\)
2. Observe \(X\).
3. Take action \(\hat{Y} = \text{BayesOptAction}(\pi Q, x)\)
4. Suffer loss \(L(\hat{Y}, Y)\)

The Goal
Minimize expected loss with respect to \(P\) not \(\pi Q\).
Meta-Bayesian Analysis

Key ideas:
- Believe \((X, Y) \sim P\)
- But predict using \(\pi Q(\cdot|X = x)\) for some prior \(\pi\)
- Prior \(\pi\) is an choice/decision/action.
- Loss associated with \(\pi\) and \((x, y)\) is

\[
L^*(\pi, (x, y)) = L(\text{BayesOptAction}(\pi Q, x), y)
\]

Meta-Bayesian risk

- Bayes risk under \(P\) of doing Bayesian analysis under \(\pi Q\)

\[
R(P, \pi) = \int L^*(\pi, (x, y))P(dx \times dy).
\]

- Meta-Bayesian optimal prior minimizes meta-Bayesian risk:

\[
\inf_{\pi \in \mathcal{F}} R(P, \pi),
\]

where \(\mathcal{F}\) is some set of priors under consideration.
Meta-Bayesian Analysis

Recipe

- Step 1: State $P$, $Q_\theta$, and select a loss function $L$;
- Step 2: Choose prior $\pi$ that minimizes meta-Bayesian risk.

Examples

- Log loss: minimizing the conditional relative entropy
  \[
  \inf_{\pi} \int KL(P^2(x, \cdot) \| \pi Q(\cdot | x)) P^1(dx)
  \]
  where $P(dx, dy) = P^1(dx) P^2(x, dy)$.

- Quadratic loss: minimizing the expected quadratic distance between two posterior means $\pi Q(\cdot | x)$ and $P^2(x, \cdot)$:
  \[
  \inf_{\pi} \int \| m_{\pi} Q(x) - m_{P^2}(x) \|_2^2 P^1(dx)
  \]
Meta-Bayesian Analysis

High-level Goals

- Meta-Bayesian analysis for $Q_\theta$ under $P$ is generally no easier than doing Bayesian analysis under $P$ directly.
- But $P$ serves only as a placeholder for an impossible-to-express true belief.
- Our theoretical approach is to attempt to prove general theorems true of broad classes of “true beliefs” $P$.
- The hope is that this will tell us something deep about subjective Bayesianism.

Remaining results are some key findings.
Meta-Bayesianism sometimes violates traditional Bayesian tenets.
Meta-Bayesian 101: if true belief is realizable

When model is well-specified

- There exists $\pi$ such that $P = \int Q_\theta \pi(d\theta)$ (i.e. $P = \pi Q$)
- Meta-Bayesian loss reduces to expected loss in traditional Bayesian
- Self-consistency: $\pi$ is the meta-Bayesian optimal prior.

Meta-Bayesian Analysis reduces to traditional Bayesian Analysis when model is well-specified.
Meta-Bayesian Analysis for i.i.d. Normal Model

Example: i.i.d. Normal

- true belief $P: \mathcal{N}(\theta, r^2)$, with $\tilde{\pi}(d\theta) \sim \mathcal{N}(0, 1)$.
- model $Q_\theta = \mathcal{N}(\theta, s^2)$ where $s^2 \neq r^2$.
- prior $\pi: \mathcal{N}(0, V)$ with one parameter $V$.
- $X \in \mathcal{R}^n$, $Y \in \mathcal{R}^k$.

Results for $n = 1$ and $k = 1$

- Predictive of $Y$ given $X = x$:
  $P: \mathcal{N}\left(\frac{x}{1+r^2}, r^2 + \frac{r^2}{1+r^2}\right)$
  $\pi Q: \mathcal{N}\left(\frac{x}{1+s^2/V}, s^2 + \frac{s^2}{1+s^2/V}\right)$
- Quadratic Loss: $V_{opt} = \frac{s^2}{r^2}$
- Log Loss: $V_{opt}$ balances predictive mean and variance.
- If well-specified ($s^2 = r^2$), $V_{opt} = 1$ for both losses.

In general, the optimal prior depends on $n$, $k$ and the loss!
General Results when $P$ is a mixture of i.i.d.

**Theorem (Berk 1966).** Posterior distribution of $\theta$ concentrates asymptotically on point minimizing the KL divergence.

\[ P = \int \tilde{P}_\psi \nu(d\psi) \]

\[ \tilde{P}_\psi \]

\[ \arg \min_{\theta} \text{KL}(\tilde{P}_\psi \| Q_\theta) \]

**Conjecture**

- For each $\psi \in \Psi$, assume there is a unique parameter $\phi(\psi) \in \Theta$ such that $Q_{\phi(\psi)}$ minimizes the KL divergence with $\tilde{P}_\psi$.
- Maybe “KL-projection” of prior, i.e., $\tilde{\pi} = \tilde{\nu} \circ \phi^{-1}$, is optimal.
General Results when $P$ is a mixture of i.i.d.

- Let $\tilde{\pi} = \tilde{\nu} \circ \phi^{-1}$ and $\tilde{\nu}(d\psi|\theta)$ be disintegration of $\tilde{\nu}$ along $\phi$.
- We can transform true model over $\Psi$ to one over $\Theta$:

$$P_\theta = \int \tilde{P}_\psi \tilde{\nu}(d\psi|\theta).$$

- Belief about first $k$ observations: $P^{(k)} = \int_\Theta P^k_\theta \tilde{\pi}(d\theta)$.

**Theorem (Y.–Roy)**

For every $\theta \in \Theta$, assume $\theta$ is the unique point in $\Theta$ achieving the infimum $\inf_{\theta' \in \Theta} KL(Q_{\theta'}||P_\theta)$. Then

$$\left| KL(P^{(k)}||\pi^*_k Q^k) - KL(P^{(k)}||\tilde{\pi} Q^k) \right| \to 0 \text{ as } k \to \infty.$$

True belief about asymptotic “location” of posterior distribution is an asymptotically optimal (surrogate) prior.
Example

data are coin tosses: 10001001100001000100100

- true belief $P$: two state $\{0, 1\}$ discrete Markov chain with transition matrix $\begin{bmatrix} 1 - p & p \\ q & 1 - q \end{bmatrix}$.

- model $Q_\theta^k = \text{Bernoulli}(\theta)^k$.

- true prior belief

$$\tilde{\nu}(dp, dq) = \tilde{\pi}(d\theta) \tilde{\kappa}(d\psi|\theta),$$

where

$$\theta = \frac{p}{p + q}$$

is the limiting relative frequency of 1’s (LRF).
What does a prior on an i.i.d. Bernoulli model mean?

Conjecture
Optimal prior for the model $Q_\theta^k$ is our true belief $\tilde{\pi}(d\theta)$ on the LRF.
In general, false!

Counterexample
Assume we know $\theta = \frac{1}{2}$.

▶ **Truth**: Sticky Markov Chain:

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0000001111111100000011111111
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▶ **Model**: i.i.d. sequence

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0010011101001011001001001001
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If we make one observation ($n = 1$) and then make one prediction ($k = 1$) better off with $Beta(0.01, 0.01)$ prior than true belief $\delta_{\frac{1}{2}}$ on LRF.
What does a prior on an i.i.d. Bernoulli model mean?

Theorem (Y.–Roy)

1. Let $Q_\theta^k$ be the i.i.d. Bernoulli model.
2. Let $P$ be true belief and assume $P$ believes in LRF.
3. Let $\tilde{\pi}(d\theta)$ be the true belief about the LRF and assume $\tilde{\pi}$ is absolutely continuous.
4. Let $\pi_k^* = \arg\min_\pi R(P, \pi)$ be an optimal surrogate prior.

Then

$$\left| \underbrace{\text{KL}(P^{(k)}||\pi_k^* Q^k)}_{R(P,\pi_k^*)} \right| - \underbrace{\text{KL}(P^{(k)}||\tilde{\pi} Q^k)}_{R(P,\tilde{\pi})} \rightarrow 0 \text{ as } k \rightarrow \infty. $$

True belief about limiting relative frequency is an asymptotically optimal (surrogate) prior.
Conclusion

▶ Standard definition of a (subjective) prior too restrictive
▶ More useful definition using Bayesian decision theory.
▶ Meta-Bayesian prior is one you believe will lead to best results.

Future Work

▶ Beyond choosing priors: General Meta-Bayesian analysis (optimal prediction algorithms)
▶ Analysis of the rationality of non-subjective procedures (e.g, switching, empirical Bayes)