On Bounding the Union Probability

Jun Yang [1]

(Joint work with Fady Alajaji[2] and Glen Takahara[2])

[1]Department of Statistical Sciences, University of Toronto, Canada

[2]Department of Mathematics and Statistics, Queen's University, Canada

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4. New Bounds using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$
5. Summary of Main Results

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Problem Formulation

- Bounds on the union probability are very useful in estimating the error probability in (coded or uncoded) communications systems. For example,

$$\max_i P(A_i) \leq P\left(\bigcup_{i=1}^{N} A_i\right) \leq \min\left\{\sum_i P(A_i), 1\right\}. \tag{1}$$

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j). \tag{2}$$

- Consider a finite family of events $A_1, \ldots, A_N$ in a finite discrete probability space $(\Omega, \mathscr{F}, P)$, where N is a fixed positive integer.
- We are interested in bounding $P\left(\bigcup_{i=1}^{N} A_i\right)$ in terms of the individual event probabilities $P(A_i)$'s and the pairwise event probabilities $P(A_i \cap A_j)$'s.

Dawson-Sankoff (DS) Bound, 1967

- For each outcome $x \in \Omega$, let the degree of x, denoted by $\deg(x)$, be the number of A_i's that contain x.
- Define $a(k) := P\left(\{x \in \bigcup_i A_i, \deg(x) = k\}\right)$, then one can verify

$$
\begin{aligned}
P\left(\bigcup_i A_i\right) &= \sum_{k=1}^{N} a(k), \\
\sum_i P(A_i) &= \sum_{k=1}^{N} k a(k), \qquad (3) \\
\sum_{i<j} P(A_i \cap A_j) &= \sum_{k=2}^{N} \binom{k}{2} a(k).
\end{aligned}
$$

Dawson-Sankoff (DS) Bound, 1967

- Using $(\theta_1, \theta_2) := \left(\sum_i P(A_i), \sum_{i<j} P(A_i \cap A_j)\right)$, the Dawson-Sankoff (DS) Bound:

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \frac{\kappa\theta_1^2}{(2-\kappa)\theta_1 + 2\theta_2} + \frac{(1-\kappa)\theta_1^2}{(1-\kappa)\theta_1 + 2\theta_2}, \tag{4}$$

where $\kappa = \frac{2\theta_2}{\theta_1} - \lfloor \frac{2\theta_2}{\theta_1} \rfloor$ and $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, is the solution of the linear programming (LP) problem:

$$\begin{aligned} \min_{\{a(k)\}} \sum_{k=1}^{N} a(k), \quad \text{s.t.} \quad & \sum_{k=1}^{N} k a(k) = \sum_i P(A_i), \\ & \sum_{k=2}^{N} \binom{k}{2} a(k) = \sum_{i<j} P(A_i \cap A_j), \\ & a(k) \geq 0, \quad k = 1, \ldots, N. \end{aligned} \tag{5}$$

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Kuai-Alajaji-Takahara (KAT) Bound, 2000

- Remind that $a(k) := P\left(\{x \in \bigcup_i A_i, \deg(x) = k\}\right)$
- Define $a_i(k) = P\left(\{x \in A_i, \deg(x) = k\right)$, one can verify

$$\begin{aligned} \sum_{i=1}^{N} a_i(k) = ka(k), \quad \Rightarrow \quad P\left(\bigcup_i A_i\right) = \sum_k a(k) = \sum_k \sum_i \frac{a_i(k)}{k}, \\ P(A_i) = \sum_{k=1}^{N} a_i(k), \quad \sum_{j:j\neq i} P(A_i \cap A_j) = \sum_{k=2}^{N} (k-1)a_i(k). \end{aligned} \tag{6}$$

- We are able to use $\left(P(A_1), \ldots, P(A_N), \sum_{j:j\neq 1} P(A_1 \cap A_j), \ldots, \sum_{j:j\neq N} P(A_N \cap A_j)\right)$.

Kuai-Alajaji-Takahara (KAT) Bound, 2000

- Let $\alpha_i := P(A_i)$, $\quad \gamma_i := \sum_j P(A_i \cap A_j) = P(A_i) + \sum_{j:j \neq i} P(A_i \cap A_j)$.
- The KAT bound is the solution of the following LP problem:

$$\min_{\{a_i(k) \geq 0\}} \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{a_i(k)}{k}, \quad \text{s.t.} \quad \sum_{k=1}^{N} a_i(k) = \alpha_i, \quad i = 1, \ldots, N, \qquad \sum_{k=1}^{N} k a_i(k) = \gamma_i, \quad i = 1, \ldots, N. \tag{7}$$

- which is given by

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \left\{ \left[\frac{1}{\lfloor \frac{\gamma_i}{\alpha_i} \rfloor} - \frac{\frac{\gamma_i}{\alpha_i} - \lfloor \frac{\gamma_i}{\alpha_i} \rfloor}{(1 + \lfloor \frac{\gamma_i}{\alpha_i} \rfloor)(\lfloor \frac{\gamma_i}{\alpha_i} \rfloor)} \right] \alpha_i \right\}, \tag{8}$$

where $\lfloor x \rfloor$ is the largest positive integer less than or equal to x.

Lower Bounds which are sharper than KAT Bound

- Recall that $a(k) := P\left(\{x \in \bigcup_i A_i, \deg(x) = k\}\right)$ and $a_i(k) = P\left(\{x \in A_i, \deg(x) = k\right)$, then we observe $a(k) \geq a_i(k)$ for all i and all k. Also, since $a(k) = \frac{\sum_j a_j(k)}{k}$, one can write

$$\frac{\sum_j a_j(k)}{k} \geq a_i(k)$$

  for all i and all k.

- As a special case for $k = N$, it reduces to

$$a_1(N) = a_2(N) = \cdots = a_N(N).$$

Optimal Lower Bound $\ell_{\text{NEW-1}}$ (in ISIT'14)

- The solution of the following LP problem:

$$\begin{aligned} \min_{\{a_i(k)\}} \; & \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{a_i(k)}{k}, \\ \text{s.t.} \quad & \sum_{k=1}^{N} a_i(k) = P(A_i), \quad i = 1, \ldots, N, \\ & \sum_{k=1}^{N} (k-1) a_i(k) = \sum_{j: j \neq i} P(A_i \cap A_j), \quad i = 1, \ldots, N, \\ & \frac{\sum_j a_j(k)}{k} \geq a_i(k), \quad i = 1, \ldots, N, \quad k = 1, \ldots, N, \\ & a_i(k) \geq 0, \quad k = 1, \ldots, N, \quad i = 1, \ldots, N. \end{aligned} \tag{9}$$

- is optimal in the class of lower bounds which are functions of $\left(P(A_1), \ldots, P(A_N), \sum_{j:j\neq 1} P(A_1 \cap A_j), \ldots, \sum_{j:j\neq N} P(A_N \cap A_j)\right)$.

Analytical Lower Bound $\ell_{\text{NEW-2}}$ (in ISIT'14)

The new analytical lower bound is the solution of the LP problem:

$$\min_{\{a_i(k)\geq 0\}} \sum_{k=1}^{N}\sum_{i=1}^{N}\frac{a_i(k)}{k}, \quad \text{s.t.} \quad \sum_{k=1}^{N} a_i(k) = P(A_i), \quad i=1,\ldots,N,$$
$$\sum_{k=1}^{N}(k-1)a_i(k) = \sum_{j:j\neq i} P(A_i \cap A_j), \quad i=1,\ldots,N, \tag{10}$$
$$a_1(N) = a_2(N) = \cdots = a_N(N).$$

The new analytical lower bound is given by

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \delta + \sum_{i=1}^{N} \left\{\left[\frac{1}{\chi(\frac{\gamma_i'}{\alpha_i'})} - \frac{\frac{\gamma_i'}{\alpha_i'} - \chi(\frac{\gamma_i'}{\alpha_i'})}{[1+\chi(\frac{\gamma_i'}{\alpha_i'})][\chi(\frac{\gamma_i'}{\alpha_i'})]}\right]\alpha_i'\right\}, \tag{11}$$

where $\delta := \{\max_i [\gamma_i - (N-1)\alpha_i]\}^+ \geq 0, \alpha_i' := \alpha_i - \delta, \gamma_i' := \gamma_i - N\delta$, and

$$\chi(x) := \begin{cases} n-1 & \text{if } x = n \text{ where } n \geq 2 \text{ is a integer} \\ \lfloor x \rfloor & \text{otherwise} \end{cases}$$

Outline

Gallot-Kounias (GK) Bound, 1968

- The GK bound is an analytical bound which fully uses $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$.
- Recently, it was re-visited by Feng-Li-Shen[1] that the GK bound can be obtained by

$$P\left(\bigcup_{i-1}^{N} A_i\right) \geq \ell_{\mathrm{GK}} = \max_{\boldsymbol{c}\in\mathbb{R}^N} \frac{\left[\sum_i c_i P(A_i)\right]^2}{\sum_i c_i \sum_k c_k P(A_i \cap A_k)}. \tag{12}$$

- Ignoring the maximization over $\boldsymbol{c}$, the RHS is a lower bound using $\sum_i c_i P(A_i)$ and $\sum_k c_k P(A_i \cap A_k)$.

[1] "Some inequalities in functional analysis, combinatorics, and probability theory", The Electronic Journal of Combinatorics, 2010.

New expressions for $P\left(\bigcup_{i=1}^N A_i\right)$

Denote $\mathscr{B}$ as the collection of all non-empty subsets of $\{1, 2, \ldots, N\}$ and let $B \in \mathscr{B}$ be a non-empty subset of $\{1, 2, \ldots, N\}$,

$$p_B := p\left(\{\omega_B, \omega_B \in A_i \text{ for all } i \in B, \omega_B \notin A_i \text{ for all } i \notin B\}\right). \tag{13}$$

Then we have a new (novel) expression of $P\left(\bigcup_{i=1}^N A_i\right)$ for any given $\boldsymbol{c}$:

$$P\left(\bigcup_{i=1}^N A_i\right) = \sum_{B \in \mathscr{B}} p_B = \sum_{i=1}^N \left(\sum_{B \in \mathscr{B}: i \in B} \frac{c_i p_B}{\sum_{k \in B} c_k}\right). \tag{14}$$

Furthermore, we have

$$\begin{aligned} P(A_i) &= \sum_{B \in \mathscr{B}: i \in B} p_B, \\ \sum_k c_k P(A_i \cap A_k) &= \sum_{k=1}^N \sum_{B: i \in B, k \in B} c_k p_B = \sum_{B: i \in B} \left(\sum_{k \in B} c_k\right) p_B. \end{aligned} \tag{15}$$

New expressions for $P\left(\bigcup_{i=1}^{N} A_i\right)$

If $\boldsymbol{c} \in \mathbb{R}_+^N$, by Cauchy-Schwarz inequality

$$P\left(\bigcup_i A_i\right) = \sum_{i=1}^{N} \left(\sum_{B \in \mathscr{B}: i \in B} \frac{c_i p_B}{\sum_{k \in B} c_k} \right) \geq \sum_{i=1}^{N} \frac{c_i^2 P(A_i)^2}{c_i \sum_k c_k P(A_i \cap A_k)}. \quad (16)$$

Note that we can use Cauchy-Schwarz Inequality again to get

$$P\left(\bigcup_i A_i\right) \geq \sum_{i=1}^{N} \frac{c_i^2 P(A_i)^2}{c_i \sum_k c_k P(A_i \cap A_k)} \geq \frac{\left[\sum_i c_i P(A_i)\right]^2}{\sum_i \sum_k c_i c_k P(A_i \cap A_k)}. \quad (17)$$

New Class of Lower Bounds $\ell_{\text{NEW-3}}(\boldsymbol{c})$

The new class of lower bounds $\ell_{\text{NEW-3}}(\boldsymbol{c})$ is given by

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \ell_i(\boldsymbol{c}) =: \ell_{\text{NEW-3}}(\boldsymbol{c}), \tag{18}$$

where

$$\begin{aligned}
\ell_i(\boldsymbol{c}) := \min_{\{p_B : i \in B\}} & \sum_{B: i \in B} \frac{c_i p_B}{\sum_{k \in B} c_k} \\
\text{s.t.} & \sum_{B: i \in B} p_B = P(A_i), \\
& \sum_{B: i \in B} \left(\frac{\sum_{k \in B} c_k}{c_i} \right) p_B = \frac{1}{c_i} \sum_k c_k P(A_i \cap A_k), \\
& p_B \geq 0, \quad \text{for all} \quad B \in \mathscr{B} \quad \text{such that} \quad i \in B.
\end{aligned} \tag{19}$$

New Class of Lower Bounds $\ell_{\text{NEW-3}}(\boldsymbol{c})$

- The solution of $\ell_i(\boldsymbol{c})$ is given by

$$\ell_i(\boldsymbol{c}) = P(A_i)\left(\frac{c_i}{\sum_{k\in B_1^{(i)}} c_k} + \frac{c_i}{\sum_{k\in B_2^{(i)}} c_k} - \frac{c_i \sum_k c_k P(A_i \cap A_k)}{P(A_i)\left(\sum_{k\in B_1^{(i)}} c_k\right)\left(\sum_{k\in B_2^{(i)}} c_k\right)}\right) \quad (20)$$

where $B_1^{(i)}$ and $B_2^{(i)}$ are subsets of $\{1,\ldots,N\}$. For $\boldsymbol{c} \in \mathbb{R}_+^N$,

$$B_1^{(i)} = \arg\max_{\{B:i\in B\}} \frac{\sum_{k\in B} c_k}{c_i} \quad \text{s.t.} \quad \frac{\sum_{k\in B} c_k}{c_i} \le \frac{\sum_k c_k P(A_i \cap A_k)}{c_i P(A_i)},$$
$$B_2^{(i)} = \arg\min_{\{B:i\in B\}} \frac{\sum_{k\in B} c_k}{c_i} \quad \text{s.t.} \quad \frac{\sum_{k\in B} c_k}{c_i} \ge \frac{\sum_k c_k P(A_i \cap A_k)}{c_i P(A_i)}, \quad (21)$$

- which are 0/1 Knapsack Problems (Pseudo-Polynomial and Polynomial-time approx).
- $\ell_{\text{NEW-3}}(\kappa\mathbf{1}) = \ell_{\text{KAT}}$ for any $\kappa \neq 0$.

Another Class of Lower Bounds $\ell_{\text{NEW-4}}(\boldsymbol{c})$ for $\boldsymbol{c} \in \mathbb{R}_+^N$

Defining $\mathscr{B}^- = \mathscr{B} \setminus \{1, \ldots, N\}$, $\tilde{\gamma}_i := \sum_k c_k P(A_i \cap A_k)$, $\tilde{\alpha}_i := P(A_i)$ and

$$\tilde{\delta} := \left[\frac{\tilde{\gamma}_i - (\sum_k c_k - \min_k c_k)\, \tilde{\alpha}_i}{\min_k c_k} \right]^+, \tag{22}$$

another class is given by $\ell_{\text{NEW-4}}(\boldsymbol{c}) := \tilde{\delta} + \sum_{i=1}^N \ell_i'(\boldsymbol{c}, \tilde{\delta})$, where

$$\ell_i'(\boldsymbol{c}, x) = [P(A_i) - x] \cdot \left(\frac{c_i}{\sum_{k \in B_1^{(i)}} c_k} + \frac{c_i}{\sum_{k \in B_2^{(i)}} c_k} - \frac{c_i \sum_k c_k [P(A_i \cap A_k) - x]}{[P(A_i) - x] \left(\sum_{k \in B_1^{(i)}} c_k \right) \left(\sum_{k \in B_2^{(i)}} c_k \right)} \right). \tag{23}$$

- $\ell_{\text{NEW-4}}(\kappa \mathbf{1}) = \ell_{\text{NEW-2}}$ for any $\kappa > 0$; $\ell_{\text{NEW-4}}(\boldsymbol{c}) \geq \ell_{\text{NEW-3}}(\boldsymbol{c})$ if $\boldsymbol{c} \in \mathbb{R}_+^N$; $\ell_{\text{NEW-4}}(\tilde{\boldsymbol{c}}) \geq \ell_{\text{NEW-3}}(\tilde{\boldsymbol{c}}) \geq \ell_{\text{GK}}$ if $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$;
- Optimal $\boldsymbol{c}$ in either class is still open.

Outline

Optimal Bounds with Exponential Complexity

- The following (exhaustive) LP problem with $2^N - 1$ number of variables gives the optimal lower/upper bound established using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$:

$$\min_{\{p_B, B \in \mathscr{B}\}} \Big/ \max_{\{p_B, B \in \mathscr{B}\}} \sum_{B \in \mathscr{B}} p_B,$$
$$\text{s.t.} \quad \sum_{i,j \in B, B \in \mathscr{B}} p_B = P(A_i \cap A_j), \quad i,j \in \{1, \ldots, N\},$$
$$p_B \geq 0, B \in \mathscr{B}. \tag{24}$$

- The optimality of (24) can be easily proved by showing its achievability: for each p_B, construct an outcome ω_B such that $p(\omega_B) = p_B$ and let $\omega_B \in A_i, \forall i \in B$.
- However, the computational complexity of the optimal lower bound (24) is exponential.

A Relaxed Problem

- Consider the following relaxed problem:

$$\min_{\{p_B, B \in \mathscr{B}\}} / \max_{\{p_B, B \in \mathscr{B}\}} \sum_{B \in \mathscr{B}} p_B,$$

$$\text{s.t.} \quad \sum_{i,j \in B, B \in \mathscr{B}} p_B = P(A_i \cap A_j), \quad i,j \in \{1, \ldots, N\},$$

$$\sum_{B: i,j,l \in B, |B|=k} p_B \geq 0, \qquad \sum_{B: i,j \in B, l \notin B, |B|=k} p_B \geq 0, \qquad (25)$$

$$\sum_{B: i \in B, j,l \notin B, |B|=k} p_B \geq 0, \qquad \sum_{B: i,j,l \notin B, |B|=k} p_B \geq 0,$$

$$\forall i,j,l,k \in \{1, \ldots, N\}.$$

- The solution of problem (25) coincides with the optimal lower/upper bound by (24) when $N \leq 7$.

The optimal feasible point of (25) is also optimal in

$$
\begin{aligned}
\min_{\{p_B, B\in\mathscr{B}\}} / \max_{\{p_B, B\in\mathscr{B}\}} & \sum_{B\in\mathscr{B}} p_B, \\
\text{s.t.} \quad & \sum_{i,j\in B, B\in\mathscr{B}} p_B = P(A_i \cap A_j), \quad i,j \in \{1,\dots,N\}, \\
& \sum_{B:i,j,l\in B,|B|=k} p_B + \sum_{B:i,j\in B, l\notin B,|B|=k} p_B \geq 0, \\
& \sum_{B:l\in B, i,j\notin B,|B|=k} p_B + \sum_{B:i,j,l\notin B,|B|=k} p_B \geq 0, \\
& \sum_{B:i,j,l\in B,|B|=k} p_B + \sum_{B:i,j,l\notin B,|B|=k} p_B \geq 0, \\
& \sum_{B:i,j\in B, l\notin B,|B|=k} p_B + \sum_{B:l\in B, i,j\notin B,|B|=k} p_B \geq 0, \\
& \sum_{B:i,j\in B,|B|=k} p_B + \sum_{B:i\in B, j,l\notin B,|B|=k} p_B \geq 0, \\
& \forall i,j,l,k \in \{1,\dots,N\}.
\end{aligned} \tag{26}
$$

New Numerical Bounds

- Define

$$a_{ij}(k) := P\left(\{x \in A_i \cap A_j, \deg(x) = k\}\right), \quad i, j, k \in \{1, \cdots, N\}. \tag{27}$$

- Consider $a_{ij}(k)$ as $\frac{(N-1)^3+N+3}{2}$ variables.
- Then $a(k)$ and $a_i(k)$ are linear functions of $a_{ij}(k)$:

$$\sum_{j=1}^{N} \frac{a_{ij}(k)}{k} = P\left(\{x \in A_i, \deg(x) = k\}\right) = a_i(k). \tag{28}$$

$$
\begin{aligned}
\min_{\{p_B, B\in\mathscr{B}\}} / \max_{\{p_B, B\in\mathscr{B}\}} \sum_{B\in\mathscr{B}} p_B &= \sum_k \sum_i \sum_j \frac{a_{ij}(k)}{k^2}, \\
\text{s.t.} \quad \sum_k a_{ij}(k) &= \sum_{i,j\in B, B\in\mathscr{B}} p_B = P(A_i \cap A_j), \quad i,j \in \{1,\dots,N\}, \\
a_{ij}(k) &= \sum_{B:i,j,l\in B, |B|=k} p_B + \sum_{B:i,j\in B, l\notin B, |B|=k} p_B \geq 0, \\
a(k) - a_i(k) - a_j(k) + a_{ij}(k) &= \sum_{B:l\in B, i,j\notin B, |B|=k} p_B + \sum_{B:i,j,l\notin B, |B|=k} p_B \geq 0, \\
a(k) - a_l(k) - a_i(k) - a_j(k) & \\
+ a_{ij}(k) + a_{il}(k) + a_{jl}(k) &= \sum_{B:i,j,l\in B, |B|=k} p_B + \sum_{B:i,j,l\notin B, |B|=k} p_B \geq 0, \\
a_l(k) + a_{ij}(k) - a_{il}(k) - a_{jl}(k) &= \sum_{B:i,j\in B, l\notin B, |B|=k} p_B + \sum_{B:l\in B, i,j\notin B, |B|=k} p_B \geq 0, \\
a_i(k) - a_{ij}(k) &= \sum_{B:i,l\in B, j\notin B, |B|=k} p_B + \sum_{B:i\in B, j,l\notin B, |B|=k} p_B \geq 0, \\
& \forall i,j,l,k \in \{1,\dots,N\}.
\end{aligned}
\tag{29}
$$

Outline

Summary of Main Results

1. In ISIT'14: Bounds using $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$
   - Optimal Numerical Bound $\ell_{\text{NEW-1}}$ (LP with $N^2 - N + 1$ variables);
   - Analytical Lower Bound $\ell_{\text{NEW-2}}$;
   - $\ell_{\text{NEW-1}} \geq \ell_{\text{NEW-2}} \geq \ell_{\text{KAT}}$.
2. New Bounds using $\{P(A_i)\}$ and $\{\sum_j c_j P(A_i \cap A_j)\}$
   - New Class of Lower Bounds $\ell_{\text{NEW-3}}(\boldsymbol{c})$ (Pseudo-polynomial if $\boldsymbol{c} \in \mathbb{R}_+^N$);
   - New Class of Lower Bounds $\ell_{\text{NEW-4}}(\boldsymbol{c})$ for $\boldsymbol{c} \in \mathbb{R}_+^N$ (Pseudo-polynomial);
   - $\ell_{\text{NEW-3}}(\kappa \mathbf{1}) = \ell_{\text{KAT}}$, $\ell_{\text{NEW-4}}(\kappa \mathbf{1}) = \ell_{\text{NEW-2}}$;
   - $\ell_{\text{NEW-4}}(\boldsymbol{c}) \geq \ell_{\text{NEW-3}}(\boldsymbol{c})$ if $\boldsymbol{c} \in \mathbb{R}_+^N$;
   - $\ell_{\text{NEW-4}}(\tilde{\boldsymbol{c}}) \geq \ell_{\text{NEW-3}}(\tilde{\boldsymbol{c}}) \geq \ell_{\text{GK}}$ if $\tilde{\boldsymbol{c}} \in \mathbb{R}_+^N$
3. New Bounds using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$
   - New Numerical Bound $\ell_{\text{NEW-5}}$ (LP with $\frac{(N-1)^3+N+3}{2}$ variables);
   - $\ell_{\text{NEW-5}} = \ell_{\text{OPT}}$ when $N \leq 7$;
   - $\ell_{\text{NEW-5}} \geq \ell_{\text{NEW-1}}$.

Comparisons of lower bounds [2]

System	I	II*	III*	IV	V	VI	VII	VIII*
N	6	6	6	7	3	4	4	4
$P\left(\bigcup_{i=1}^{N} A_i\right)$	0.7890	0.6740	0.7890	0.9687	0.3900	0.3252	0.5346	0.5854
KAT	0.7247	0.6227	0.7222	0.8909	0.3833	0.2769	0.4434	0.5412
GK	0.7601	0.6510	0.7508	0.9231	0.3813	0.2972	0.4750	0.5390
$\ell_{\text{NEW-2}}$	0.7247	0.6227	0.7222	0.8909	**0.3900**	0.3205	0.4562	0.5464
$\ell_{\text{NEW-1}}$	0.7487	0.6398	0.7427	0.9044	**0.3900**	**0.3252**	**0.5090**	0.5531
$\ell_{\text{NEW-4}}$ $(\tilde{\boldsymbol{c}}^{+})$	0.7638	0.6517	0.7512	0.9231	**0.3900**	0.2951	0.4905	0.5412
$\ell_{\text{NEW-4}}$ (rd)	0.7783	0.6633	0.7810	0.9501	**0.3900**	0.3203	0.4992	0.5666
$\ell_{\text{NEW-5}}$	**0.7890**	**0.6740**	**0.7890**	**0.9687**	**0.3900**	**0.3252**	**0.5090**	**0.5673**

[2] In the table, * indicates $\tilde{\boldsymbol{c}} \in \mathbb{R}_{+}^{N}$ and a bold number indicates coincidence with the optimal bound (24).

References

- $\ell_{\text{NEW-1}}$ and $\ell_{\text{NEW-2}}$:
  [1] J. Yang, F. Alajaji, and G. Takahara, Lower bounds on the probability of a finite union of events, http://arxiv.org/abs/1401.5543
  [2] —, New bounds on the probability of a finite union of events, ISIT'14.
- $\ell_{\text{NEW-4}}$ and $\ell_{\text{NEW-5}}$:
  [3] —, On Bounding the Union Probability, ISIT'15.

Thank you!