Interactions and Factorial ANOVA

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Interactions

- Interaction between explanatory variables means "It depends."
- Relationship between one explanatory variable and the response variable *depends* on the value of the other explanatory variable.
- Can have
  - Quantitative by quantitative
  - Quantitative by categorical
  - Categorical by categorical

Quantitative by Quantitative

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \epsilon$$
$$E(Y|\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$$

For fixed x_2

$$E(Y|\mathbf{x}) = (\beta_0 + \beta_2 x_2) + (\beta_1 + \beta_3 x_2)x_1$$

Both slope and intercept depend on value of x_2

And for fixed x_1, slope and intercept relating x_2 to $E(Y)$ depend on the value of x_1

Quantitative by Categorical

- One regression line for each category.
- Interaction means slopes are not equal
- Form a product of quantitative variable by each dummy variable for the categorical variable
- For example, three treatments and one covariate: x_1 is the covariate and x_2, x_3 are dummy variables

$$
\begin{aligned}
Y \;=\; & \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \\
& + \beta_4 x_1 x_2 + \beta_5 x_1 x_3 + \epsilon
\end{aligned}
$$

General principle

- Interaction between A and B means
  - Relationship of A to Y depends on value of B
  - Relationship of B to Y depends on value of A
- The two statements are formally equivalent

Make a table

$$E(Y|\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_1 x_2 + \beta_5 x_1 x_3$$

Group	x_2	x_3	$E(Y|\mathbf{x})$
1	1	0	$(\beta_0 + \beta_2) + (\beta_1 + \beta_4)x_1$
2	0	1	$(\beta_0 + \beta_3) + (\beta_1 + \beta_5)x_1$
3	0	0	$\beta_0 \quad + \quad \beta_1 \quad x_1$

Group	x_2	x_3	$E(Y|\mathbf{x})$
1	1	0	$(\beta_0 + \beta_2) + (\beta_1 + \beta_4)x_1$
2	0	1	$(\beta_0 + \beta_3) + (\beta_1 + \beta_5)x_1$
3	0	0	$\beta_0 \quad + \quad \beta_1 \quad x_1$

What null hypothesis would you test for

- Equal slopes
- Comparing slopes for group one vs three
- Comparing slopes for group one vs two
- Equal regressions
- Interaction between group and x_1

What to do if H_0: $\beta_4=\beta_5=0$ is rejected

- How do you test Group "controlling" for x_1?
- A reasonable choice is to set x_1 to its sample mean, and compare treatments at that point.

Categorical by Categorical

- Naturally part of factorial ANOVA in experimental studies
- Also applies to purely observational data

Factorial ANOVA

More than one categorical
explanatory variable

Factorial ANOVA

- Categorical explanatory variables are called **factors**

- More than one at a time

- Primarily for true experiments, but also used with observational data

- If there are observations at all combinations of explanatory variable values, it's called a *complete* factorial design (as opposed to a fractional factorial).

The potato study

- Cases are potatoes
- Inoculate with bacteria, store for a fixed time period.
- Response variable is percent surface area with visible rot.
- Two explanatory variables, randomly assigned
  - Bacteria Type (1, 2, 3)
  - Temperature (1=Cool, 2=Warm)

Two-factor design

	Bacteria Type		
Temp	1	2	3
1=Cool			
2=Warm			

Six treatment conditions

Factorial experiments

- Allow more than one factor to be investigated in the same study: Efficiency!

- Allow the scientist to see whether the effect of an explanatory variable *depends* on the value of another explanatory variable: Interactions

- Thank you again, Mr. Fisher.

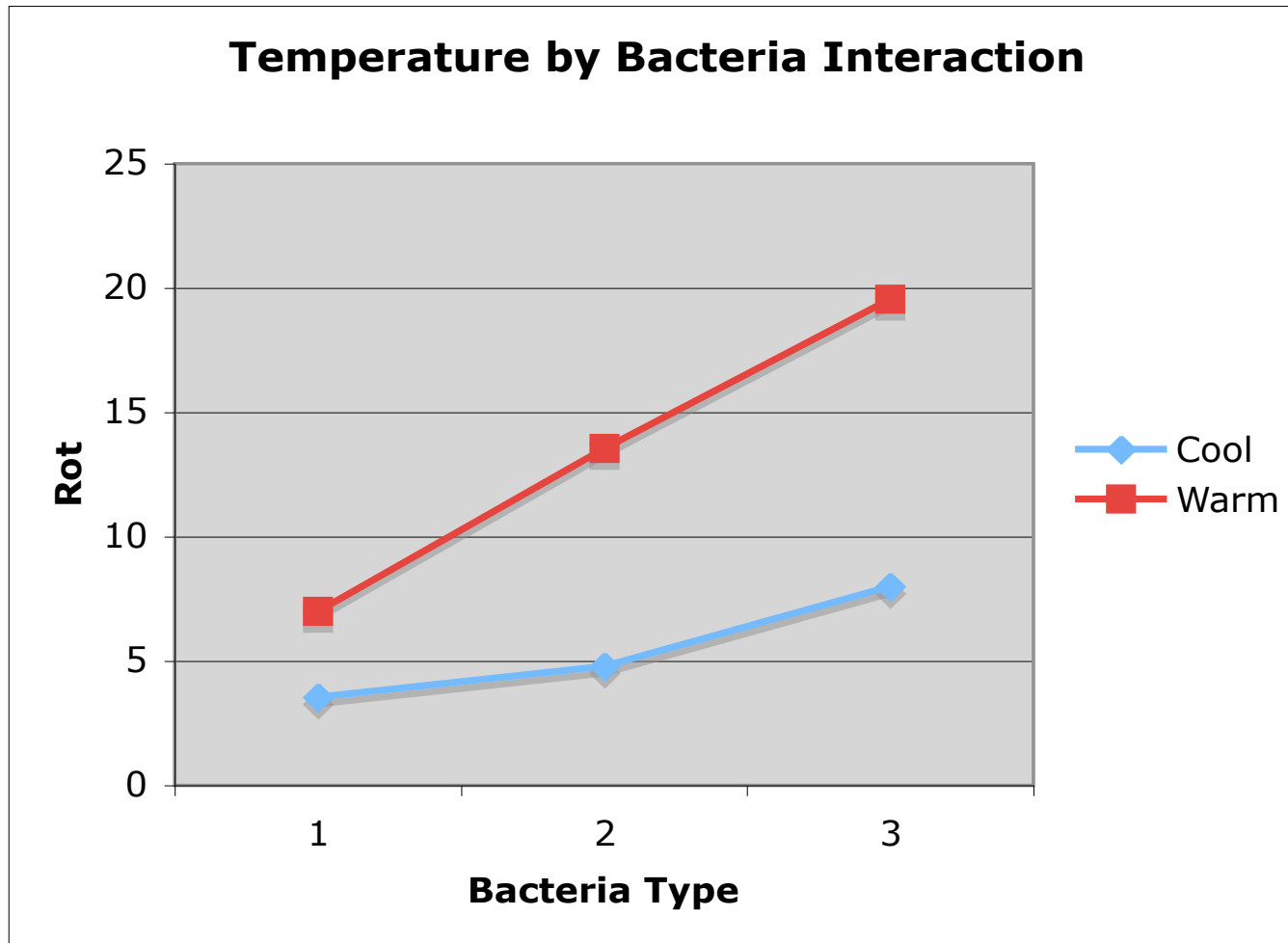# Normal with equal variance and conditional (cell) means $\mu_{i,j}$

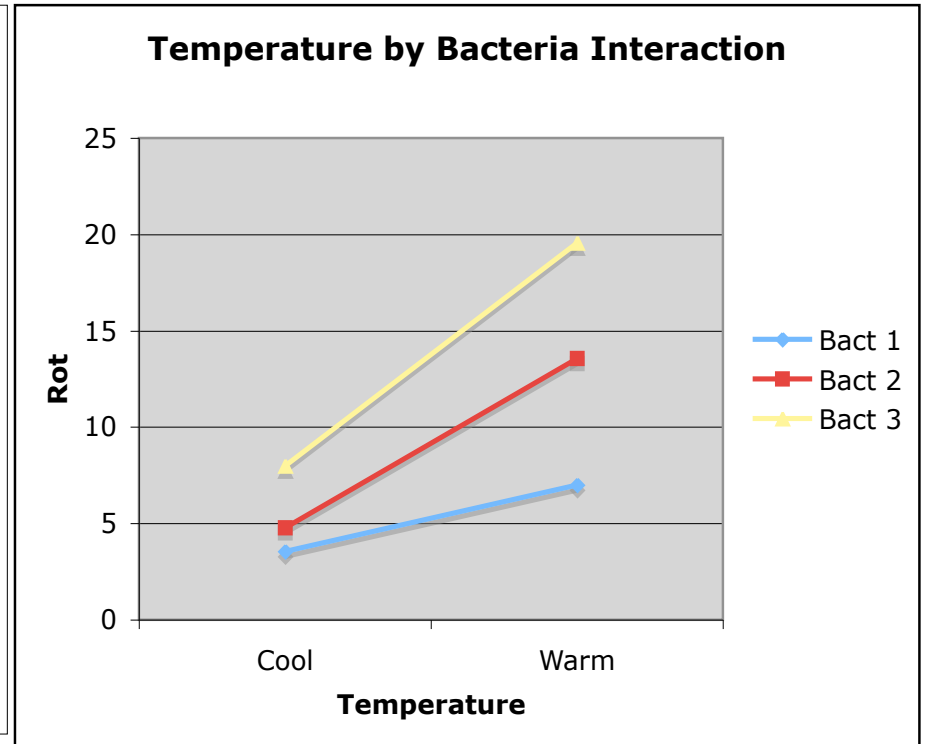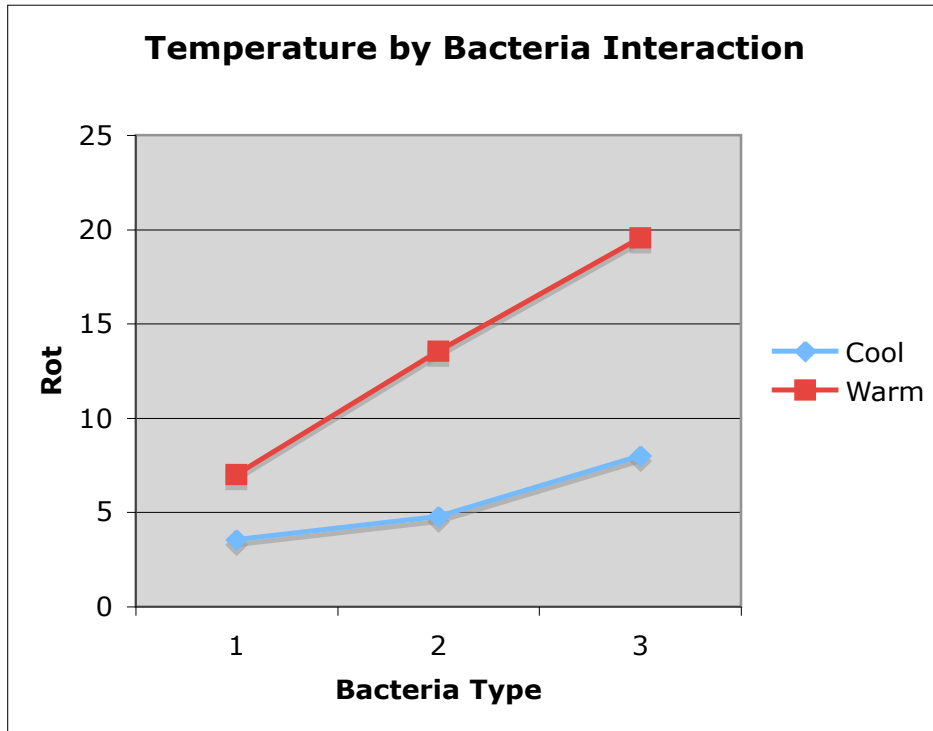|	Bacteria Type			
Temp	1	2	3	
1=Cool	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{1,3}$	$\dfrac{\mu_{1,1} + \mu_{1,2} + \mu_{1,3}}{3}$
2=Warm	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{2,3}$	$\dfrac{\mu_{2,1} + \mu_{2,2} + \mu_{2,3}}{3}$
	$\dfrac{\mu_{1,1} + \mu_{2,1}}{2}$	$\dfrac{\mu_{1,2} + \mu_{2,2}}{2}$	$\dfrac{\mu_{1,3} + \mu_{2,3}}{2}$	μ

Tests

- Main effects: Differences among marginal means

- Interactions: Differences between differences (What is the effect of Factor A? **It depends** on the level of Factor B.)
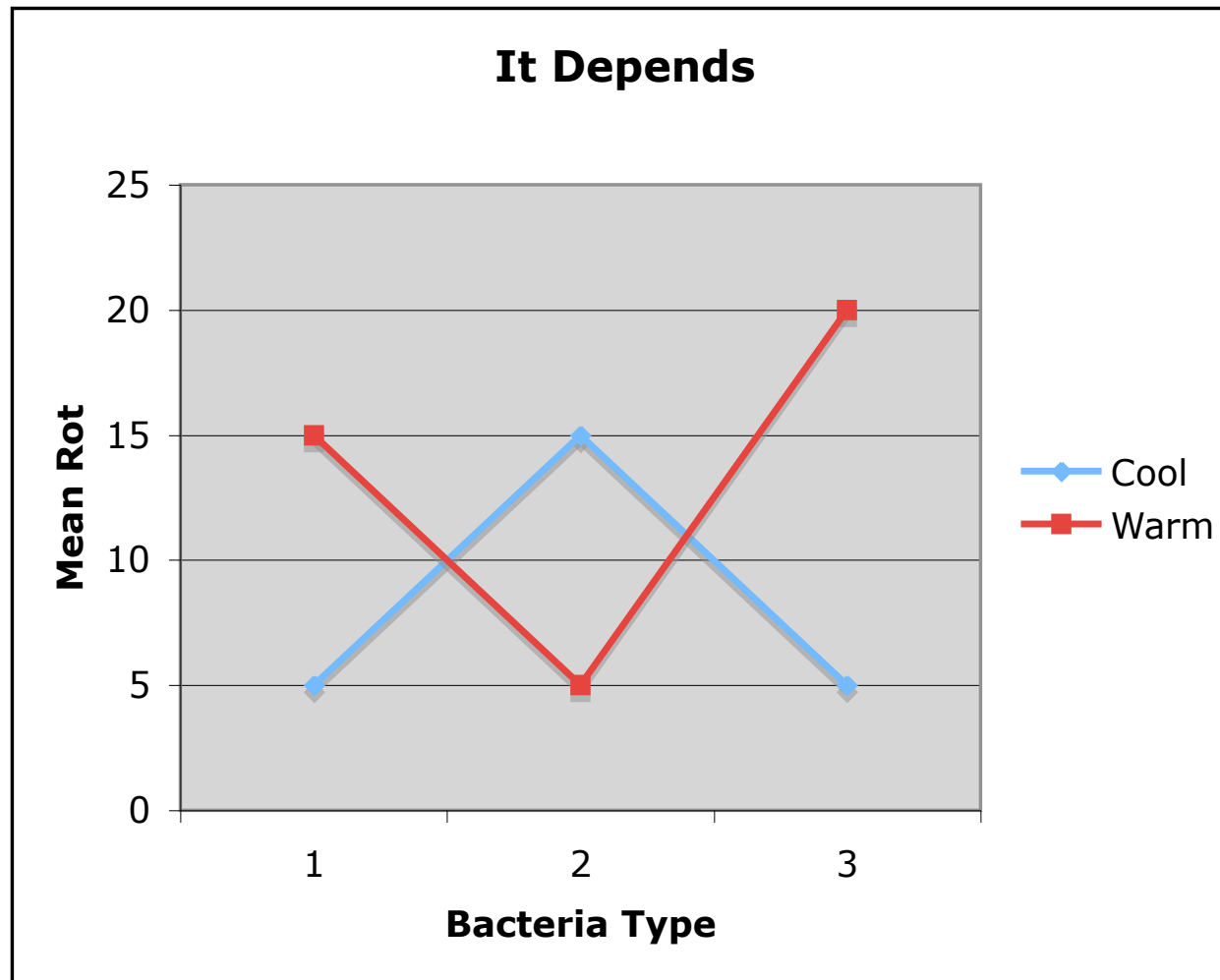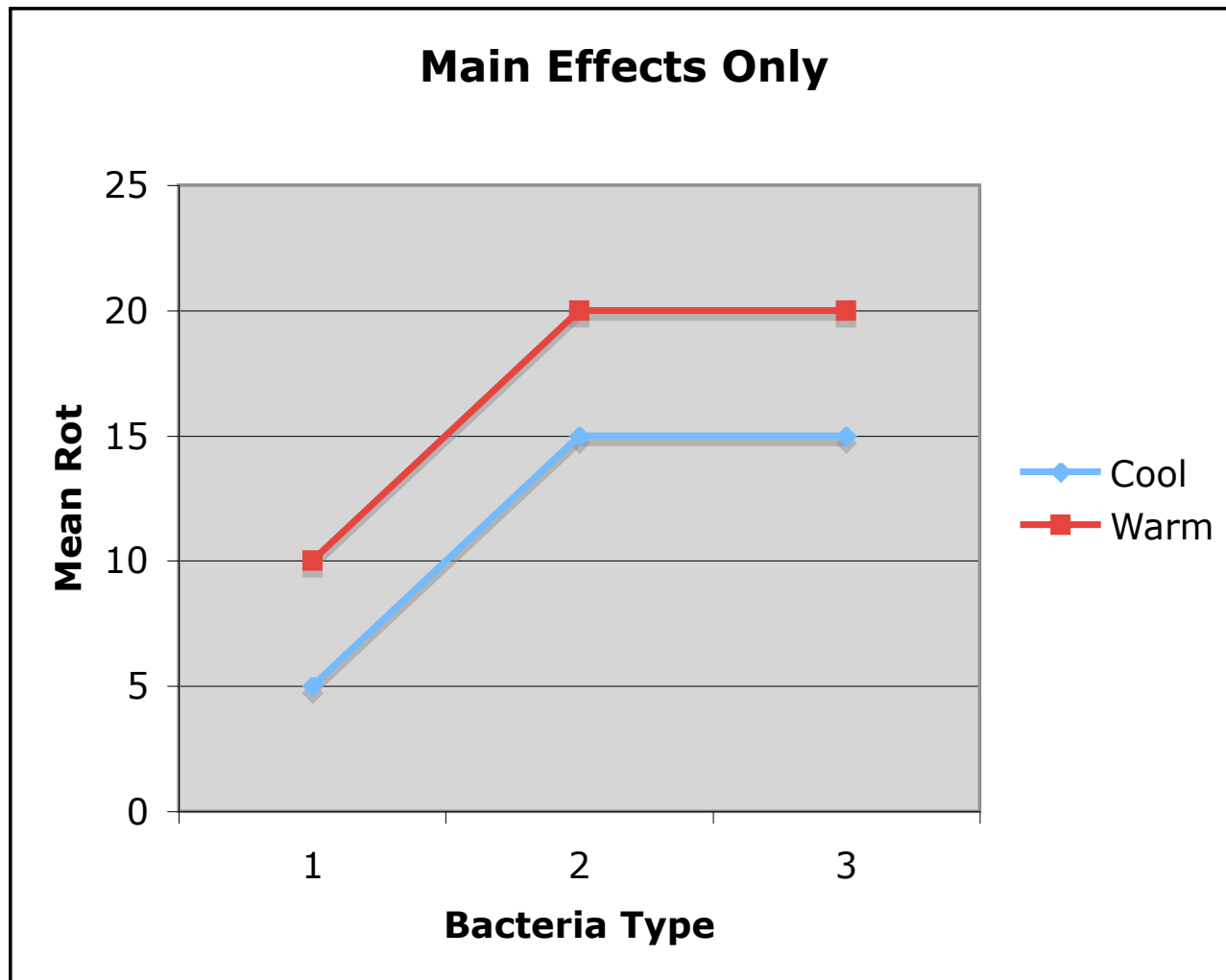
To understand the interaction, plot the means

Either Way
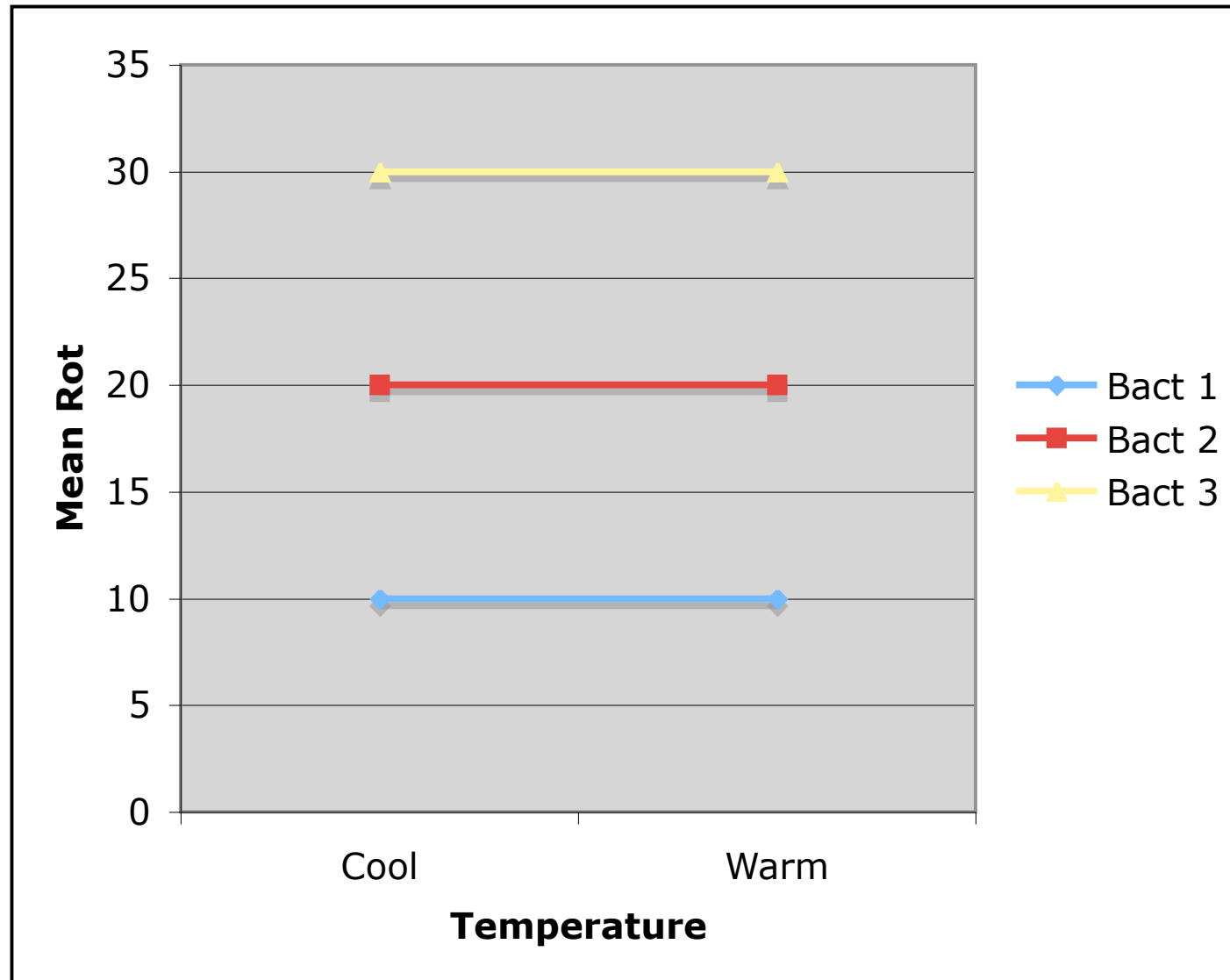
Non-parallel profiles = Interaction

It Depends

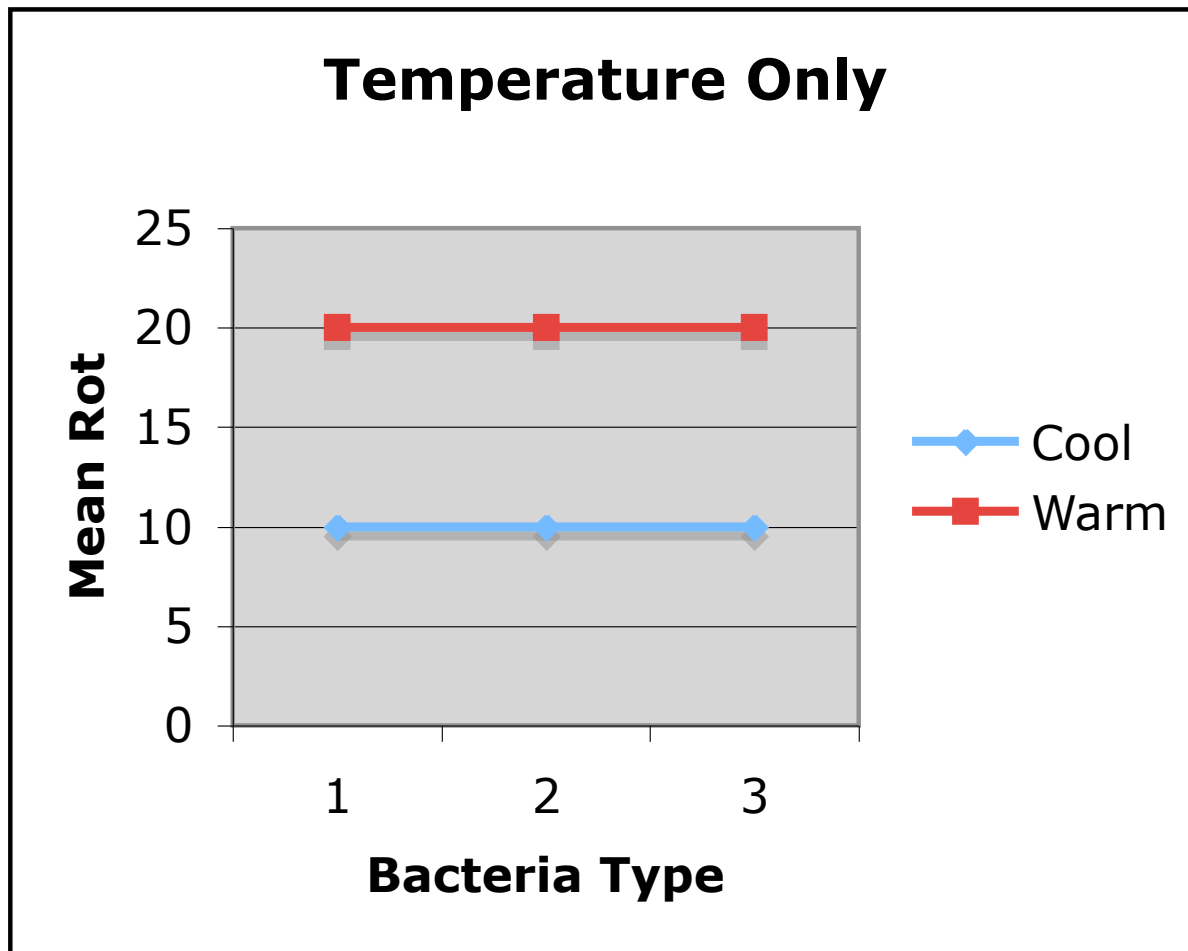(Chart: Mean Rot vs Bacteria Type, with series "Cool" and "Warm")

Main effects for both variables, no interaction

Main effect for Bacteria only

Main Effect for Temperature Only

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Both Main Effects, and the Interaction

Mean Rot as a Function of Temperature and Bacteria Type
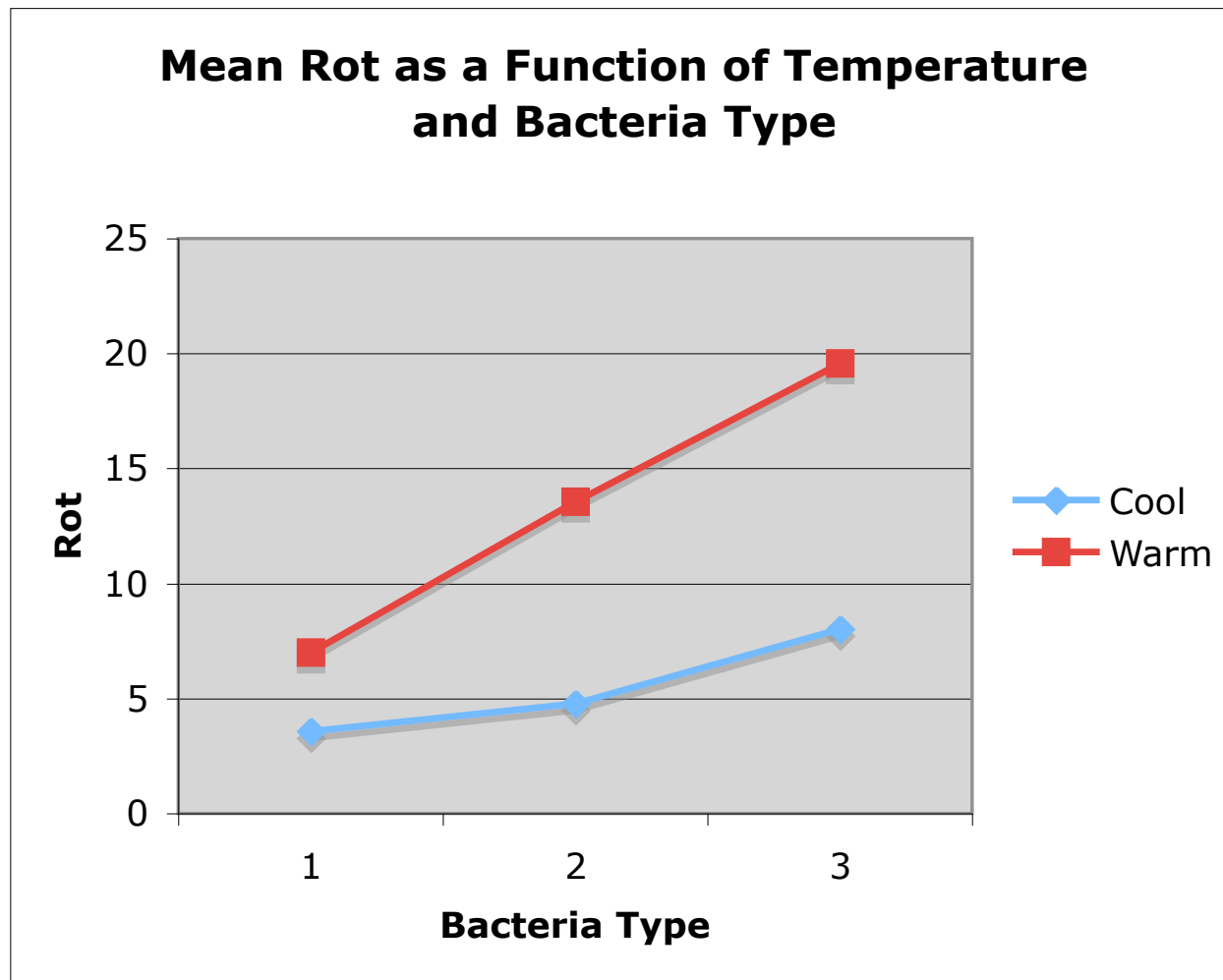
Should you interpret the main effects?
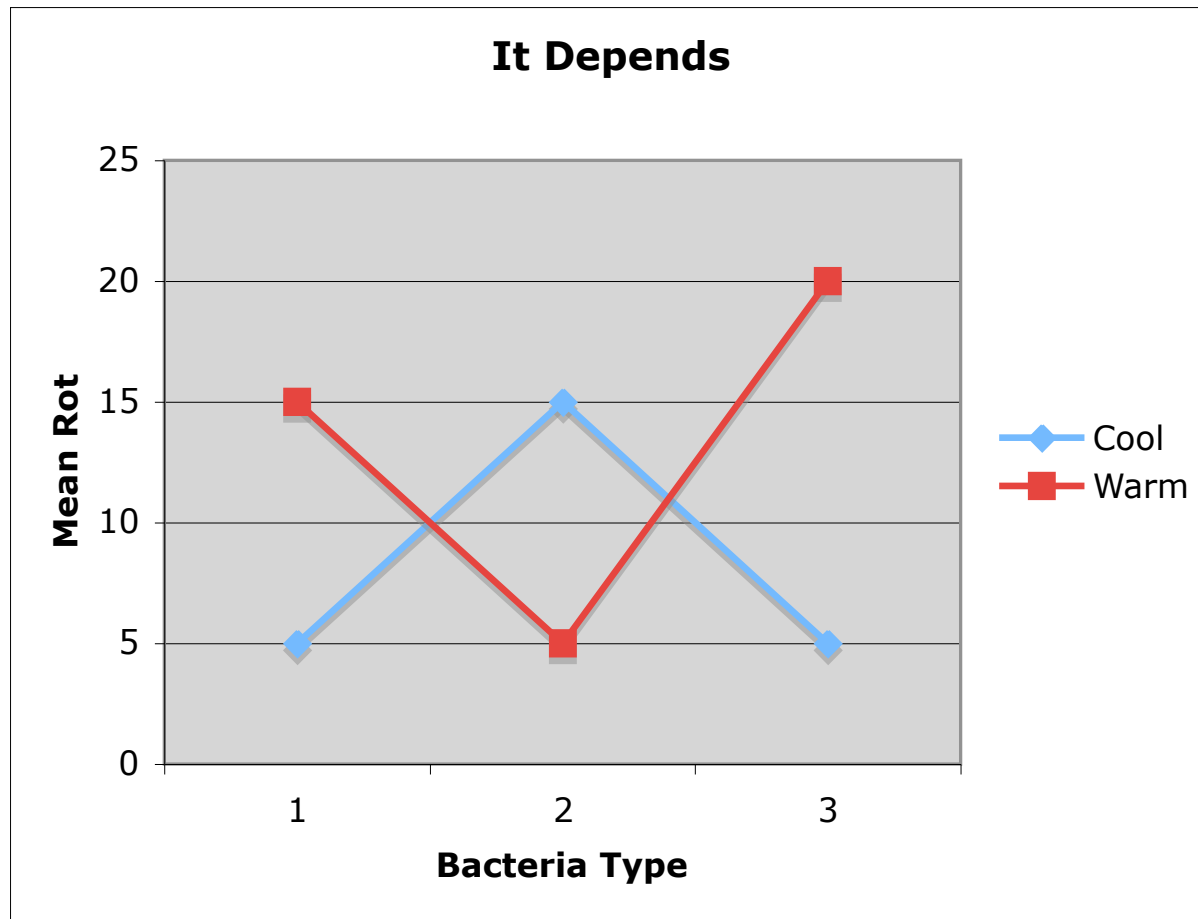
A common error

- Categorical explanatory variable with p categories
- p dummy variables (rather than $p-1$)
- And an intercept

- There are p population means represented by $p+1$ regression coefficients - not unique

But suppose you leave off the intercept

- Now there are p regression coefficients and p population means

- The correspondence is unique, and the model can be handy -- less algebra

- Called **cell means coding**

Cell means coding: p indicators and no intercept

$$E[Y|\boldsymbol{X} = \boldsymbol{x}] = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

Drug	x_1	x_2	x_3	$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$
A	1	0	0	$\mu_1 = \beta_1$
B	0	1	0	$\mu_2 = \beta_2$
Placebo	0	0	1	$\mu_3 = \beta_3$

Add a covariate: x_4

$$E[Y|\boldsymbol{X} = \boldsymbol{x}] = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4$$

Drug	x_1	x_2	x_3	$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4$
A	1	0	0	$\beta_1 + \beta_4 x_4$
B	0	1	0	$\beta_2 + \beta_4 x_4$
Placebo	0	0	1	$\beta_3 + \beta_4 x_4$

Contrasts

$$c = a_1 \mu_1 + a_2 \mu_2 + \cdots + a_p \mu_p$$

$$\widehat{c} = a_1 \overline{Y}_1 + a_2 \overline{Y}_2 + \cdots + a_p \overline{Y}_p$$

where $a_1 + a_2 + \cdots + a_p = 0$

In a one-factor design

- Mostly, what you want are tests of contrasts,
- Or collections of contrasts.
- You could do it with any dummy variable coding scheme.
- Cell means coding is often most convenient.
- With $\boldsymbol{\beta}=\boldsymbol{\mu}$, test H_0: $\mathbf{L}\boldsymbol{\beta}=\mathbf{h}$

- Can get a confidence interval for any single contrast using the t distribution.

Testing Contrasts in Factorial Designs

Temp	Bacteria Type			
	1	2	3	
1=Cool	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{1,3}$	$\frac{\mu_{1,1} + \mu_{1,2} + \mu_{1,3}}{3}$
2=Warm	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{2,3}$	$\frac{\mu_{2,1} + \mu_{2,2} + \mu_{2,3}}{3}$
	$\frac{\mu_{1,1} + \mu_{2,1}}{2}$	$\frac{\mu_{1,2} + \mu_{2,2}}{2}$	$\frac{\mu_{1,3} + \mu_{2,3}}{2}$	μ

- **Differences between marginal means are definitely contrasts**

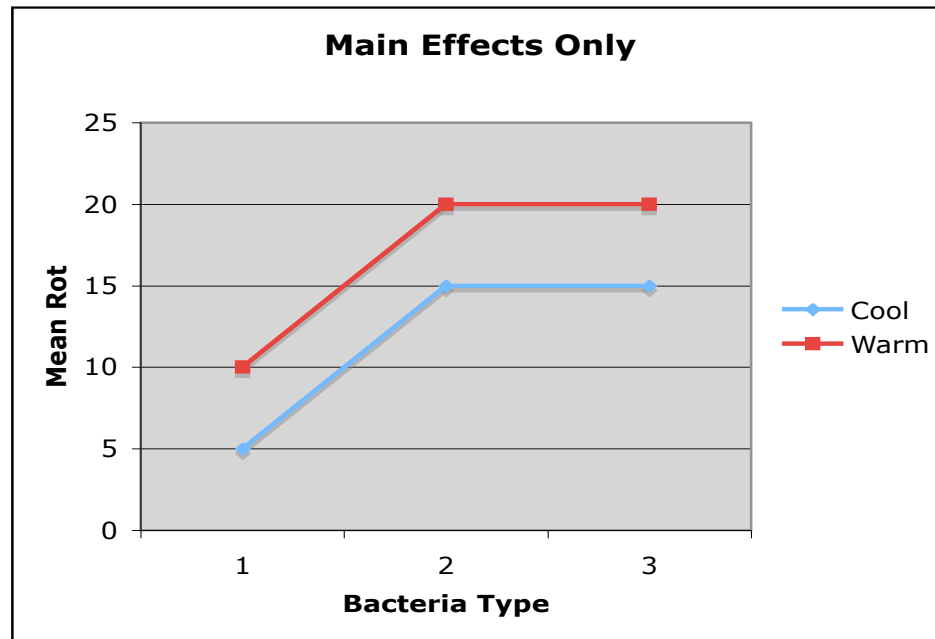- **Interactions are also sets of contrasts**

Interactions are sets of Contrasts

Temp	Bacteria Type			
	1	2	3	
1=Cool	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{1,3}$	$\frac{\mu_{1,1}+\mu_{1,2}+\mu_{1,3}}{3}$
2=Warm	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{2,3}$	$\frac{\mu_{2,1}+\mu_{2,2}+\mu_{2,3}}{3}$
	$\frac{\mu_{1,1}+\mu_{2,1}}{2}$	$\frac{\mu_{1,2}+\mu_{2,2}}{2}$	$\frac{\mu_{1,3}+\mu_{2,3}}{2}$	μ

- $H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$

- $H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1}$ and

  $\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$

Interactions are sets of Contrasts

- $H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$

- $H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1}$ and

  $\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$

Equivalent statements

- The effect of A depends upon B
- The effect of B depends on A

$$H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$$

$$H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1} \text{ and}$$
$$\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$$

Three factors: A, B and C

- There are three (sets of) main effects: One each for A, B, C
- There are three two-factor interactions
    - A by B (Averaging over C)
    - A by C (Averaging over B)
    - B by C (Averaging over A)
- There is one three-factor interaction: AxBxC

Meaning of the 3-factor interaction

- The form of the A x B interaction depends on the value of C

- The form of the A x C interaction depends on the value of B

- The form of the B x C interaction depends on the value of A

- These statements are equivalent. Use the one that is easiest to understand.

To graph a three-factor interaction

- Make a separate mean plot (showing a 2-factor interaction) for each value of the third variable.

- In the potato study, a graph for each type of potato

Four-factor design

- Four ~sets of~ main effects
- Six two-factor interactions
- Four three-factor interactions
- One four-factor interaction: The nature of the three-factor interaction depends on the value of the 4th factor
- There is an F test for each one
- And so on …

As the number of factors increases

- The higher-way interactions get harder and harder to understand
- All the tests are still tests of sets of contrasts (differences between differences of differences …)
- But it gets harder and harder to write down the contrasts
- Effect coding becomes easier

Effect coding

Like indicator dummy variables with intercept, but put -1
for the last category.

Bact	B_1	B_2
1	1	0
2	0	1
3	-1	-1

Temperature	T
1=Cool	1
2=Warm	-1

$$E(Y|\mathbf{X} = \mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

Interaction effects are products of dummy variables

$$E(Y|\mathbf{X} = \mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

- The A x B interaction: Multiply each dummy variable for A by each dummy variable for B
- Use these products as additional explanatory variables in the multiple regression
- The A x B x C interaction: Multiply each dummy variable for C by each product term from the A x B interaction
- Test the sets of product terms simultaneously

Make a table

$$E(Y|\mathbf{X} = \mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

Bact	Temp	B_1	B_2	T	B_1T	B_2T	$E(Y|\mathbf{X} = \mathbf{x})$
1	1	1	0	1	1	0	$\beta_0 + \beta_1 + \beta_3 + \beta_4$
1	2	1	0	-1	-1	0	$\beta_0 + \beta_1 - \beta_3 - \beta_4$
2	1	0	1	1	0	1	$\beta_0 + \beta_2 + \beta_3 + \beta_5$
2	2	0	1	-1	0	-1	$\beta_0 + \beta_2 - \beta_3 - \beta_5$
3	1	-1	-1	1	-1	-1	$\beta_0 - \beta_1 - \beta_2 + \beta_3 - \beta_4 - \beta_5$
3	2	-1	-1	-1	1	1	$\beta_0 - \beta_1 - \beta_2 - \beta_3 + \beta_4 + \beta_5$

Cell and Marginal Means

Tmp	Bacteria Type			
	1	2	3	
1=C	$\beta_0 + \beta_1 + \beta_3 + \beta_4$	$\beta_0 + \beta_2 + \beta_3 + \beta_5$	$\beta_0 - \beta_1 - \beta_2 + \beta_3 - \beta_4 - \beta_5$	$\beta_0 + \beta_3$
2=W	$\beta_0 + \beta_1 - \beta_3 - \beta_4$	$\beta_0 + \beta_2 - \beta_3 - \beta_5$	$\beta_0 - \beta_1 - \beta_2 - \beta_3 + \beta_4 + \beta_5$	$\beta_0 - \beta_3$
	$\beta_0 + \beta_1$	$\beta_0 + \beta_2$	$\beta_0 - \beta_1 - \beta_2$	β_0

We see

- Intercept is the grand mean
- Regression coefficients for the dummy variables are deviations of the marginal means from the grand mean
- What about the interactions?

$$E(Y|\mathbf{X} = \mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

A bit of algebra shows

$$\mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} \text{ is equivalent to } \beta_4 = \beta_5$$

$$\mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3} \text{ is equivalent to } \beta_4 = -\beta_5$$

$$\text{So } \beta_4 = \beta_5 = 0$$

Factorial ANOVA with effect coding is pretty automatic

- You don't have to make a table unless asked.

- It always works as you expect it will.

- Hypothesis tests are the same as testing sets of contrasts.

- Covariates present no problem. Main effects and interactions have their usual meanings, "controlling" for the covariates.

- Plot the "least squares means" (Y-hat at x-bar values for covariates).

Again

- Intercept is the grand mean
- Regression coefficients for the dummy variables are deviations of the marginal means from the grand mean
- Test of main effect(s) is test of the dummy variables for a factor.
- Interaction effects are products of dummy variables.

Balanced vs. Unbalanced Experimental Designs

- Balanced design: Cell sample sizes are proportional (maybe equal)

- Explanatory variables have zero relationship to one another

- Numerator SS in ANOVA are independent

- Everything is nice and simple

- Most experimental studies are designed this way.

- As soon as somebody drops a test tube, it's no longer true

Analysis of unbalanced data

- When explanatory variables are related, there is potential ambiguity.

- A is related to Y, B is related to Y, and A is related to B.

- Who gets credit for the portion of variation in Y that could be explained by either A or B?

- With a regression approach, whether you use contrasts or dummy variables (equivalent), the answer is **nobody**.

- Think of full, reduced models.

- Equivalently, general linear test

Some software is designed for balanced data

- The special purpose formulas are much simpler.

- They were very useful *in the past*.

- Since most data are at least a little unbalanced, thy are a recipe for trouble.

- Most textbook data are balanced, so they cannot tell you what your software is really doing.

- R's `anova` and `aov` functions are designed for balanced data, though `anova` applied to lm objects can give you what you want if you use it with care.

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