Analysis of Fractional Factorial Designs[1]

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Fractional Factorial Designs

- So far, we have considered only *complete factorials*.
- In a complete factorial, there are observations at all treatment combinations.
- In a fractional factorial, some cells in the design are deliberately empty.
- Why? Usually expense.

Models for fractional factorial designs

- You can still fit a regression model if you are willing to make some assumptions.
- Usually, assume one or more interactions are absent.
- Its another example of the tradeoff between assumptions and amount of data.
- The more data you have, the less you have to assume.

The simplest example: Two by two

Omit the red cell

	B = Yes	B = No
A = Yes	μ_{11}	μ_{12}
A = No	μ_{21}	μ_{22}

No interaction means the effect of A is the same for both levels of B. $\mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} \Leftrightarrow \mu_{22} = \mu_{12} - \mu_{11} + \mu_{21}$
And the difference between marginal means for A is

$$
\begin{aligned}
& \frac{1}{2}(\mu_{11} + \mu_{12}) - \frac{1}{2}(\mu_{21} + \mu_{22}) \\
= & \frac{1}{2}\left(\mu_{11} + \mu_{12} - \mu_{21} - (\mu_{12} - \mu_{11} + \mu_{21})\right) \\
= & \frac{1}{2}\left(\mu_{11} + \mu_{12} - \mu_{21} - \mu_{12} + \mu_{11} - \mu_{21}\right) \\
= & \frac{1}{2}\left(2\mu_{11} - 2\mu_{21}\right) \\
= & \mu_{11} - \mu_{21}
\end{aligned}
$$

Extensions

- In a $2 \times 2 \times \cdots \times 2$ factorial, You can sacrifice any cell you want in exchange for the highest-way interaction.
- Chapter 6A in Cochran and Cox's *Design of experiments* has a lot of rules that apply to balanced designs.
- Here's another approach.

For larger designs

- All the standard tests are tests of whether contrasts or collections of contrasts equal zero.
- You can sacrifice any contrast in exchange for a cell by
  - Choosing one of the μ parameters involved in the contrast.
  - Solving for it.
  - Letting that cell be empty.
- You can do this for more than one contrast (and cell).
- How do you know what contrasts to test for the remaining effects?
- Substitute the solution(s) for the μ parameter(s).
- Calculate the contrast you would usually test.
- And simplify.
- Just as in the 2×2 example.
- The hardest part is knowing what contrasts correspond to an effect of interest for larger designs.
- There is a systematic way to find out.

Effect coding

- Pick an interaction or set of interactions to sacrifice.
- The number of potential empty cells equals the number of βs set to zero.
- Each β is zero if and only if a linear combination of the μ values is zero.
- It's a matter of going back and forth between cell means coding and effect coding.
- To get an explicit formula for the β parameters of effect coding in terms of the μ parameters of cell means coding.

Example: Crop yield study

Three Fertilizers by Sprinkler versus Drip Irrigation

$$E[Y|\mathbf{X}] = \beta_0 + \beta_1 f_1 + \beta_2 f_2 + \beta_3 w + \beta_4 f_1 w + \beta_5 f_2 w$$

Fertilizer	Water	f_1	f_2	w	$f_1 w$	$f_2 w$	$E[Y\|\mathbf{X}]$
1	Sprinkler	1	0	1	1	0	$\mu_{11} = \beta_0 + \beta_1 + \beta_3 + \beta_4$
1	Drip	1	0	-1	-1	0	$\mu_{12} = \beta_0 + \beta_1 - \beta_3 - \beta_4$
2	Sprinkler	0	1	1	0	1	$\mu_{21} = \beta_0 + \beta_2 + \beta_3 + \beta_5$
2	Drip	0	1	-1	0	-1	$\mu_{22} = \beta_0 + \beta_2 - \beta_3 - \beta_5$
3	Sprinkler	-1	-1	1	-1	-1	$\mu_{31} = \beta_0 - \beta_1 - \beta_2 + \beta_3 - \beta_4 - \beta_5$
3	Drip	-1	-1	-1	1	1	$\mu_{32} = \beta_0 - \beta_1 - \beta_2 - \beta_3 + \beta_4 + \beta_5$

- The μ_{ij} are linear combinations of the β_j.
- And the coefficients are sitting right there in the table.

Matrix form

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} = \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \\ \mu_{22} \\ \mu_{31} \\ \mu_{32} \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}\boldsymbol{\beta} &= \boldsymbol{\mu} \\ \boldsymbol{\beta} &= \mathbf{A}^{-1}\boldsymbol{\mu} \end{aligned}$$

This is really nice because it shows the equivalence of the two dummy variable coding schemes.

Can even do most of the job with R

$\boldsymbol{\beta} = \mathbf{A}^{-1}\boldsymbol{\mu}$

```
> A = rbind(  c(1, 1, 0, 1, 1, 0),
+             c(1, 1, 0,-1,-1, 0),
+             c(1, 0, 1, 1, 0, 1),
+             c(1, 0, 1,-1, 0,-1),
+             c(1,-1,-1, 1,-1,-1),
+             c(1,-1,-1,-1, 1, 1) )
> solve(A) # Inverse
           [,1]       [,2]       [,3]       [,4]       [,5]       [,6]
[1,]  0.1666667  0.1666667  0.1666667  0.1666667  0.1666667  0.1666667
[2,]  0.3333333  0.3333333 -0.1666667 -0.1666667 -0.1666667 -0.1666667
[3,] -0.1666667 -0.1666667  0.3333333  0.3333333 -0.1666667 -0.1666667
[4,]  0.1666667 -0.1666667  0.1666667 -0.1666667  0.1666667 -0.1666667
[5,]  0.3333333 -0.3333333 -0.1666667  0.1666667 -0.1666667  0.1666667
[6,] -0.1666667  0.1666667  0.3333333 -0.3333333 -0.1666667  0.1666667
> 0.1666667 * 6
[1] 1
```

- This identifies the linear combination of μs that correspond to each β.
- Still have to solve for the cell mean you're omitting, and substitute.
- But at least now we know what linear combinations to calculate.

Which cells can we omit?

And still be able to test the remaining effects

- Try omitting one or more cells.
- Solve for that μ in terms of the other μs.
- Substitute the solution for the missing cell mean(s).
- Set the contrast(s) you want the test to zero (get these from $\mathbf{A}^{-1}$)
- Simplify.
- If you get $0 = 0$, you've omitted the wrong cells.
- Otherwise, you know what special hypotheses to test.

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