STA 260s20 Assignment Six: Hypothesis Testing Part One[1]

Please read Sections 6.3.3-6.3.6 in the text, pages 332-344. Notice how the authors like to go from the testing problem directly to the p-value, with the test statistic an intermediate step that is sometimes not even identified as such. Rather than comparing the test statistic to a critical value, they just compare the p-value to α. Their emphasis on the normal model with known variance (the "location normal model") is helpful for understanding even though it is never used in practice.

The following homework problems are not to be handed in. They are preparation for Quiz 6 (Week of March 2nd) and Term Test 2. **Please try each question before looking at the solution**. Use the formula sheet.

1. On Test Two and the final exam, you may be asked for some well-known distribution facts. You are also responsible for the proofs if requested, but in these questions you just write the answer from memory. You need to have these things in your head in order to put important derivations together. This is largely a repeat from Assignment Four.

   (a) Let $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, where a and b are constants. What is the distribution of Y?

   (b) Let $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$. What is the distribution of Z?

   (c) Let $Z \sim N(0, 1)$. What is the distribution of $Y = Z^2$?

   (d) Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of $Y = \sum_{i=1}^n X_i$?

   (e) Let $X_1, \ldots, X_n$ be independent random variables, with $X_i \sim N(\mu_i, \sigma_i^2)$. Let $a_1, \ldots, a_n$ be constants. What is the distribution of $Y = \sum_{i=1}^n a_i X_i$?

   (f) Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of the sample mean $\overline{X}_n$?

   (g) Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of $\frac{\sqrt{n}(\overline{X}-\mu)}{\sigma}$? Are you using the Central Limit Theorem?

   (h) Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of $\frac{(n-1)S^2}{\sigma^2}$?

   (i) Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of $\frac{\sqrt{n}(\overline{X}-\mu)}{S}$?

   (j) Let $X_1, \ldots, X_n$ be independent $\chi^2(\nu_i)$ random variables. What is the distribution of $Y = \sum_{i=1}^n X_i$?

[1] Copyright information is at the end of the last page.

2. Vocabulary is important too. You need to know what the words mean in order to understand the lectures in this class and later statistics classes. Questions like these may be on Test Two and the final exam. I am even asking about Type I and Type II error; as long as you are memorizing things, you might as well know this arbitrary designation too. Fill in the blanks.

   (a) A collection of independent and identically distributed random variables is called a _______.

   (b) A function of the sample data that is not a function of any unknown parameters is called a _______

   (c) An estimator T_n is said to be _______ for θ if $E(T_n) = \theta$.

   (d) An estimator T_n is said to be _______ for θ if $T_n \xrightarrow{p} \theta$.

   (e) Quantities such as $E(X)$, $Var(X)$, $E(X^k)$, $E(X^2Y^2)$ and so on are called (population) _______.

   (f) The sample moment corresponding to $E(X^4)$ is _______.

   (g) The set of values that can be taken on by a parameter or parameter vector is called the _______.

   (h) The vector of observed data values is a point in the _______.

   (i) A statement like $\theta \in \Omega_0$ is called the _______.

   (j) A statement like $\theta \in \Omega_1$ is called the _______.

   (k) Ideas like that nothing is happening, the treatment had no effect, it makes no difference, no action is required, and so on are expressed by the _______.

   (l) The null hypothesis is rejected when the data vector falls into the _______.

   (m) Often, the critical region is defined in terms of the value of a _______ statistic.

   (n) H_0 is rejected when the test statistic is beyond some particular number. That number is called a _______.

   (o) Failure to reject the null hypothesis when the null hypothesis is false is called a _______.

   (p) Rejection of the null hypothesis when the null hypothesis is true is called a _______.

   (q) If the critical region of a test is denoted by C, then $\max_{\theta \in \Omega_0} P_\theta(X \in C)$ is called the _______ of the test, and is usually denoted by the Greek letter ___.

   (r) The maximum probability of rejecting H_0 when H_0 is true is called the _______ of the test.

   (s) The "size" of a test is another term for the _______.

   (t) The probability of correctly rejecting the null hypothesis is called the _______ of the test.

3. Do Exercise 6.3.1, except do it this way.

   (a) Write down a formula for the test statistic.

   (b) What is the distribution of the test statistic? Is the distribution exact, or is it asymptotic?

   (c) For $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$ with significance level α (which is what the authors intend in the question),

      i. What is the set Ω_0?
      ii. What is the set Ω_1?
      iii. Give the critical value(s).
      iv. Give the critical value(s) for $\alpha = 0.05$. The answer is numerical.
      v. Give the critical value(s) for $\alpha = 0.01$. The answer is numerical.
      vi. What is the decision rule? That is, when will the null hypothesis be rejected?
      vii. Calculate the test statistic. Show some work. The answer is a number. Circle your answer.
      viii. Do you reject the null hypothesis at $\alpha = 0.05$? Answer Yes or No.
      ix. Calculate the p-value. The answer is a number.
      x. What do you conclude? Choose one of these answers.
         - $\mu > 5$
         - $\mu < 5$
         - $\mu = 5$
      xi. The question asks for a 95% confidence interval too, so you may as well do it.

   (d) For $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$ with significance level α,

      i. What is the set Ω_0?
      ii. What is the set Ω_1?
      iii. Give the critical value(s).
      iv. Give the critical value(s) for $\alpha = 0.05$. The answer is numerical.
      v. Give the critical value(s) for $\alpha = 0.01$. The answer is numerical.
      vi. What is the decision rule? That is, when will the null hypothesis be rejected?
      vii. You have already calculated the test statistic. Do you reject the null hypothesis at $\alpha = 0.05$? Answer Yes or No.
      viii. Calculate the p-value. The answer is a number.
      ix. What do you conclude? Choose one of these answers.
         - $\mu \leq 5$
         - $\mu > 5$
      x. To show that the test really has significance level α, you need to prove that $\max_{\mu \in \Omega_0} P_\mu(Z \geq z_{1-\alpha})$ occurs at $\mu = \mu_0$. Do it.

xi. Derive a general formula for the power of this test. Use $\Phi(\cdot)$ to denote the cumulative distribution function of a standard normal. (This is standard notation.)

(e) For $n = 10$ and $\alpha = 0.05$ as in the problem, find the power of the test when the true value of μ is 5.5. The answer is a number. Also find the power of the test when the true value of μ is 6.

(f) For $H_0 : \mu \geq \mu_0$ versus $H_1 : \mu < \mu_0$ with significance level α,

i. What is the set Ω_0?

ii. What is the set Ω_1?

iii. Give the critical value(s).

iv. Give the critical value(s) for $\alpha = 0.05$. The answer is numerical.

v. Give the critical value(s) for $\alpha = 0.01$. The answer is numerical.

vi. What is the decision rule? That is, when will the null hypothesis be rejected?

vii. Show that $\max_{\mu\in\Omega_0} P_\mu(Z \leq z_{1-\alpha})$ occurs at $\mu = \mu_0$.

viii. You have already calculated the test statistic. Do you reject the null hypothesis at $\alpha = 0.05$? Answer Yes or No.

ix. Calculate the p-value. The answer is a number.

x. What do you conclude? Choose one of these answers.

- $\mu \geq 5$
- $\mu < 5$

4. Do Exercise 6.3.2 in a bit more detail, this way.

(a) Write down a formula for the test statistic.

(b) What is the distribution of the test statistic under the null hypothesis – that is, when H_0 is true?

(c) For $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ with significance level α (which is what the authors intend in the question),

i. What is the set Ω_0?

ii. What is the set Ω_1?

iii. Give the critical value(s).

iv. Give the critical value(s) for $\alpha = 0.05$. The answer is numerical.

v. Give the critical value(s) for $\alpha = 0.01$. The answer is numerical.

vi. What is the decision rule? That is, when will the null hypothesis be rejected?

vii. Calculate the test statistic. Show some work. The answer is a number. Circle your answer.

viii. Do you reject the null hypothesis at $\alpha = 0.05$? Answer Yes or No.

ix. What do you conclude? Choose one of these answers.
   - $\mu > 5$
   - $\mu < 5$
   - $\mu = 5$

x. The question asks for a 95% confidence interval too, so you may as well do it.

5. Do Exercise 6.3.4 in a bit more detail, this way. Note that you can't calculate the p-value without using software, so the question as it is written is impossible for a test or exam.

   (a) Write down a formula for the test statistic.

   (b) What is the distribution of the test statistic under the null hypothesis?

   (c) For $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$ with significance level α (which is what the authors intend in the question),

      i. What is the set Ω_0?

      ii. What is the set Ω_1?

      iii. Give the critical value(s).

      iv. Give the critical value(s) for $\alpha = 0.05$. The answer is numerical.

      v. Give the critical value(s) for $\alpha = 0.01$. The answer is numerical.

      vi. What is the decision rule? That is, when will the null hypothesis be rejected?

      vii. Calculate the test statistic. Show some work. The answer is a number. Circle your answer.

      viii. Do you reject the null hypothesis at $\alpha = 0.05$? Answer Yes or No.

      ix. What do you conclude? Choose one of these answers.
         - $\mu > 60$
         - $\mu < 60$
         - $\mu = 60$

      x. The question asks for a 95% confidence interval too, so you may as well do it.

6. Do Exercise 6.3.5, as written.

7. This is an adaptation of Question 3 on Assignment 3. The label on the peanut butter jar says peanuts, partially hydrogenated peanut oil, salt and sugar. But we all know there is other stuff in there too. There is very good reason to assume that the number of rat hairs in a jar of peanut butter has a Poisson distribution with mean λ, because it's easy to justify a Poisson process model for how the hairs get into the jars (technical details omitted). There is a government standard that says the expected number of rat hairs in a jar can be no more than 8. A sample of thirty 500g jars yields $\overline{X}_n = 9.2$.

   (a) What null hypothesis should be tested to decide whether the company is in compliance with regulations?

(b) What is the alternative hypothesis?

(c) Suggest *two* possible test statistics. Remember, all we know is the sample size and the sample mean.

(d) What is the distribution of the two test statistics under the null hypothesis?

(e) What is the set Ω_0?

(f) What is the set Ω_1?

(g) Give the critical value(s).

(h) Give the critical value(s) for $\alpha = 0.05$. The answer is numerical.

(i) Give the critical value(s) for $\alpha = 0.01$. The answer is numerical.

(j) What is the decision rule? That is, when will the null hypothesis be rejected?

(k) Calculate the values of *both* test statistics. Show some work. The answers are numbers.

(l) Calculate both p-values. Show some work. The answers are numbers.

(m) With each test, do you reject the null hypothesis at $\alpha = 0.05$? Answer Yes or No.

(n) What do you conclude? Choose one of these answers.

- $\lambda > 8$
- $\lambda \leq 8$

(o) Is there evidence that the company is in violation of the regulations? Answer Yes or No.

8. Do Exercise 6.3.11. In later classes (not this one), you might be expected to supply questions like the following on your own, and answer them.

(a) Suggest a model for the sample data.

(b) What null hypothesis should be tested to decide whether the die is biased? I suggest a two-tailed test.

(c) What is the alternative hypothesis?

(d) Suggest *two* possible test statistics.

(e) What is the distribution of the two test statistics under the null hypothesis?

(f) What is the set Ω_0?

(g) What is the set Ω_1?

(h) Give the critical value(s).

(i) Give the critical value(s) for $\alpha = 0.05$. The answer is numerical.

(j) Give the critical value(s) for $\alpha = 0.01$. The answer is numerical.

(k) What is the decision rule? That is, when will the null hypothesis be rejected?

(l) Calculate the values of *both* test statistics. Show some work. The answers are numbers.

(m) Calculate both p-values. Show some work. The answers are numbers.

(n) With each test, do you reject the null hypothesis at $\alpha = 0.05$? Answer Yes or No.

(o) What do you conclude for each test? Choose one of these answers.

- $\theta > \frac{1}{6}$.
- $\theta < \frac{1}{6}$.
- $\theta = \frac{1}{6}$.

(p) Is there evidence that the die is biased?

9. If the following example seems familiar, it's because it comes from Question 5 of Assignment Four. Two surgeons in a cosmetic surgery practice decide to have a friendly competition. The wait list has 20 patients who want surgery to make their noses smaller. Ten patients are randomly assigned to Surgeon A, and the other ten are assigned to Surgeon B. A panel of medical students rate the facial appearance of the patients on a 100 point scale before surgery and again six weeks after. The number for each patient is improvement (according to the medical students): After minus before. Of course, the medical students are not told which doctor did the surgery.

   Because of scheduling problems and drop-out (people change their minds), Surgeon A only did nine surgeries, and Surgeon B did seven. So with $n_1 = 9$ and $n_2 = 7$, we have $\overline{x} = 14.1$, $s_1^2 = 48.2$, $\overline{y} = 13.3$, $s_1^2 = 32.7$.

(a) State the model for this question. Given what we've done in this course so far, there is really only one choice.

(b) How could the Central Limit Theorem be used to justify a normal model for the attractiveness ratings?

(c) What is the parameter space for this model?

(d) What null hypothesis should be tested to decide which surgeon won the contest?

(e) What is the alternative hypothesis?

(f) In Question 5 of Assignment Four, you derived the test statistic, which may also be found on the formula sheet. What is its distribution under the null hypothesis?

(g) What is the set Ω_0?

(h) What is the set Ω_1?

(i) Give the critical value(s) of the test statistic.

(j) Give the critical value(s) for $\alpha = 0.05$. The answer is numerical.

(k) Give the critical value(s) for $\alpha = 0.01$. The answer is numerical.

(l) What is the decision rule? That is, when will the null hypothesis be rejected?

(m) Calculate the numerical value of the test statistic. Show your work. The answer is a number.

(n) Do you reject the null hypothesis at $\alpha = 0.05$? Answer Yes or No.

(o) What do you conclude? Choose one of these answers.

- $\mu_1 > \mu_2$.
- $\mu_1 < \mu_2$.
- $\mu_1 = \mu_2$.

(p) Who won the contest?

10. Suppose you want to test $H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_0 : \sigma_1^2 \neq \sigma_2^2$.

(a) Write down a formula for the test statistic. Look at your answer to Question 6 of Assignment 4 if you need to.

(b) What is the distribution of the test statistic under the null hypothesis?

(c) Prove your answer to the last part. Don't forget to say why numerator and denominator are independent.

(d) For the data of Question 9, calculate the test statistic. Show a little work. The answer is a number.

(e) Find the critical values for $\alpha = 0.05$ in the tables of the F distribution in the back of the text. It's not part of the formula sheet. If I ask this on the test or final exam, I might have to give you just the correct page, and you find the critical value(s).

(f) What is the decision rule?

(g) Do you reject the null hypothesis? Answer Yes or No.

(h) What do you conclude? Pick one.

- $\sigma_1^2 > \sigma_2^2$.
- $\sigma_1^2 < \sigma_2^2$.
- $\sigma_1^2 = \sigma_2^2$.

11. You have done this problem quite a few times now. This is the last time you will have to do it for homework in this class. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x)$ and cumulative distribution function $F(x)$.

(a) Let $Y_1 = \max(X_i)$.

i. Derive the cumulative distribution function of Y_1.

ii. Derive the probability density function of Y_1.

(b) Let $Y_2 = \min(X_i)$.

i. Derive the cumulative distribution function of Y_2.

ii. Derive the probability density function of Y_2.

Notice that since $f(x)$ includes an indictor for the support, the minimum and maximum both have the same support as the original distribution. This makes sense.

12. Let $X_1, \ldots, X_n$ be a random sample from an Exponential(λ) distribution, and we seek to test hypotheses about λ. In lecture, you saw a couple of test statistics based on $\overline{X}_n$. In this question, the test statistic will be $Y = \max(X_i)$. As in lecture, the focus will be on $H_0 : \lambda \geq \lambda_0$ versus $H_1 : \lambda < \lambda_0$. Again, since $E(X_i) = \frac{1}{\lambda}$, long average wait times correspond to small values of λ, and if the average wait time were under two months, then it would be surprising to get a maximum wait time that was very large, like six years. Thus, H_0 will be rejected if $Y \geq k$, for some well-chosen k.

    (a) Write a formula for the cumulative distribution function of the test statistic Y. Use indicator functions.

    (b) Give a formula for $P_\lambda(Y \geq k)$, where $k > 0$.

    (c) Show that the maximum probability of rejecting H_0 when H_0 is true is attained when $\lambda = \lambda_0$.

    (d) Determine the critical value k so that the test will have significance level α. Derive an explicit formula. Show your work.

    (e) Give the critical value for testing $H_0 : \lambda \geq \frac{1}{2}$ versus $H_1 : \lambda < \frac{1}{2}$ with $\alpha = 0.05$ and $n = 29$. The answer is a number.

    (f) A random sample of size $n = 29$ yields a maximum value of $y = 10.25$.

        i. What is the p-value? The answer is a number. Show your work.

        ii. There are two ways to decide whether to reject the null hypothesis? What are they?

        iii. Do you reject the null hypothesis? Answer Yes or No.

        iv. What do you conclude? Pick one.

            - $E(X_i) \leq 2$.
            - $E(X_i) > 2$.

    (g) Suppose that the true value of λ is 0.4, so that the null hypothesis is false in this particular way. We wonder about the probability of correctly rejecting the null hypothesis. Still with $\alpha = 0.05$,

        i. What is the power of the test with $n = 29$ (the sample size we actually had)? The answer is a number between zero and one.

        ii. What is the power of the test with $n = 290$? The answer is a number between zero and one.

13. Suppose we have two independent random samples from exponential distributions. Suggest an exact F-test of $H_0 : \lambda_1 = \lambda_2$ based on the sample means. Use page 3 of the new formula sheet.
