STA 260s20 Assignment Five: Method of Moments, Least Squares and Maximum Likelihood[1]

These homework problems are not to be handed in. They are preparation for Quiz 5 (Week of Feb. 24) and Term Test 2. **Please try each question before looking at the solution**.

In preparation for this assignment, you can actually look at the text, starting with Chapter 6. Read pages 297-302. You can skip Section 6.1.1 (bottom of page 302 to page 308. Later in the course, we will use a different and more traditional definition of a sufficient statistic. Read Section 6.2 (bottom of page 308 to page 317).

I have noticed that a major obstacle for many students when doing maximum likelihood calculations is a set of basic mathematical operations they actually know. But the mechanics are rusty, or the notation used in Statistics is troublesome. So, with sincere apologies to those who don't need this, here are some basic rules.

- The distributive law: $a(b+c) = ab + ac$. You may see this in a form like
$$\theta \sum_{i=1}^n x_i = \sum_{i=1}^n \theta x_i$$
- Power of a product is the product of powers: $(ab)^c = a^c\, b^c$. You may see this in a form like
$$\left(\prod_{i=1}^n x_i \right)^\alpha = \prod_{i=1}^n x_i^\alpha$$
- Multiplication is addition of exponents: $a^b a^c = a^{b+c}$. You may see this in a form like
$$\prod_{i=1}^n \theta e^{-\theta x_i} = \theta^n \exp(-\theta \sum_{i=1}^n x_i)$$
- Powering is multiplication of exponents: $(a^b)^c = a^{bc}$. You may see this in a form like
$$(e^{\mu t + \frac{1}{2}\sigma^2 t^2})^n = e^{n\mu t + \frac{1}{2} n \sigma^2 t^2}$$
- Log of a product is sum of logs: $\ln(ab) = \ln(a) + \ln(b)$. You may see this in a form like
$$\ln \prod_{i=1}^n x_i = \sum_{i=1}^n \ln x_i$$
- Log of a power is the exponent times the log: $\ln(a^b) = b\, \ln(a)$. You may see this in a form like
$$\ln(\theta^n) = n \ln \theta$$
- The log is the inverse of the exponential function: $\ln(e^a) = a$. You may see this in a form like
$$\ln \left(\theta^n \exp(-\theta \sum_{i=1}^n x_i) \right) = n \ln \theta - \theta \sum_{i=1}^n x_i$$

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1. Choose the correct answer.

   (a) $\prod_{i=1}^n e^{x_i} =$
      i. $\exp(\prod_{i=1}^n x_i)$
      ii. e^{nx_i}
      iii. $\exp(\sum_{i=1}^n x_i)$

   (b) $\prod_{i=1}^n \lambda e^{-\lambda x_i} =$
      i. $\lambda e^{-\lambda^n x_i}$
      ii. $\lambda^n e^{-\lambda n x_i}$
      iii. $\lambda^n \exp(-\lambda \sum_{i=1}^n x_i)$
      iv. $\lambda^n \exp(-n\lambda \sum_{i=1}^n x_i)$
      v. $\lambda^n \exp(-\lambda^n \sum_{i=1}^n x_i)$

   (c) $\prod_{i=1}^n a_i^b =$
      i. na_i^b
      ii. a_i^{nb}
      iii. $(\prod_{i=1}^n a_i)^b$

   (d) $\prod_{i=1}^n a^{b_i} =$
      i. na^{b_i}
      ii. a^{nb_i}
      iii. $\sum_{i=1}^n a^{b_i}$
      iv. $a^{\prod_{i=1}^n b_i}$
      v. $a^{\sum_{i=1}^n b_i}$

   (e) $\left(e^{\lambda(e^t-1)}\right)^n =$
      i. $ne^{\lambda(e^t-1)}$
      ii. $e^{n\lambda(e^t-1)}$
      iii. $e^{\lambda(e^{nt}-1)}$
      iv. $e^{n\lambda(e^t-n)}$

   (f) $\left(\prod_{i=1}^n e^{-\lambda x_i}\right)^2 =$
      i. $e^{-2n\lambda x_i}$
      ii. $e^{-2\lambda \sum_{i=1}^n x_i}$
      iii. $2e^{-\lambda \sum_{i=1}^n x_i}$

2. True, or False?

   (a) $\sum_{i=1}^n \frac{1}{x_i} = \frac{1}{\sum_{i=1}^n x_i}$
   (b) $\prod_{i=1}^n \frac{1}{x_i} = \frac{1}{\prod_{i=1}^n x_i}$
   (c) $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$
   (d) $\ln(a+b) = \ln(a) + \ln(b)$
   (e) $e^{a+b} = e^a + e^b$
   (f) $e^{a+b} = e^a e^b$
   (g) $e^{ab} = e^a e^b$
   (h) $\prod_{i=1}^n (x_i + y_i) = \prod_{i=1}^n x_i + \prod_{i=1}^n y_i$
   (i) $\ln(\prod_{i=1}^n a_i^b) = b\sum_{i=1}^n \ln(a_i)$
   (j) $\sum_{i=1}^n \prod_{j=1}^n a_j = n\prod_{j=1}^n a_j$
   (k) $\sum_{i=1}^n \prod_{j=1}^n a_i = \sum_{i=1}^n a_i^n$
   (l) $\sum_{i=1}^n \prod_{j=1}^n a_{i,j} = \prod_{j=1}^n \sum_{i=1}^n a_{i,j}$

3. Simplify as much as possible.

   (a) $\ln \prod_{i=1}^n \theta^{x_i}(1-\theta)^{1-x_i}$
   (b) $\ln \prod_{i=1}^n \binom{m}{x_i}\theta^{x_i}(1-\theta)^{m-x_i}$
   (c) $\ln \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$
   (d) $\ln \prod_{i=1}^n \theta(1-\theta)^{x_i-1}$
   (e) $\ln \prod_{i=1}^n \frac{1}{\theta}e^{-x_i/\theta}$
   (f) $\ln \prod_{i=1}^n \frac{1}{\beta^\alpha\Gamma(\alpha)}e^{-x_i/\beta}x_i^{\alpha-1}$
   (g) $\ln \prod_{i=1}^n \frac{1}{2^{\nu/2}\Gamma(\nu/2)}e^{-x_i/2}x_i^{\nu/2-1}$
   (h) $\ln \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$
   (i) $\prod_{i=1}^n \frac{1}{\beta-\alpha}I(\alpha \le x_i \le \beta)$ (Express in terms of the minimum and maximum y_1 and y_n.)

4. Let $X_1, \ldots, X_n$ be a random sample (that is, independent and identically distributed) from a Poisson distribution with parameter $\lambda > 0$. The sample mean for a sample of $n = 49$ is $\overline{x} = 4.2$.

   (a) Derive a formula for $\widehat{\lambda}$, the maximum likelihood estimate of λ.
   (b) Carry out the second derivative test.
   (c) Give a point estimate of λ. Your answer is a number.
   (d) Give a 95% confidence interval for λ. The answer is a pair of numbers. My lower confidence limit is 3.63.

5. Do Exercises 6.2.2 and 6.2.3 in the text.

6. (a) Do Exercise 6.2.5 in the text. Note that α_0 is known.

   (b) Suppose $\alpha_0 = 5$. Calculate your estimate for the following data. The answer is a number. Circle your answer. `Data: 6.51 3.09 2.87 1.35`

7. Let $X_1, \ldots, X_n$ be a random sample from a distribution with probability mass function $p(x|\theta) = \theta(1-\theta)^{x-1} I(x = 1, 2, \ldots)$, where $0 < \theta < 1$.

   (a) Derive a general expression for the Maximum Likelihood Estimator (MLE). Carry out the second derivative test to make sure you really have a maximum. Circle your final answer.

   (b) Use these data to calculate a numerical estimate: `5, 1, 2, 1, 2, 4, 3, 17, 4, 1, 5, 4, 7, 17, 1, 1, 2, 2, 7, 11`. Answer: 0.2061856

8. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\theta) = \frac{1}{\theta} e^{-x/\theta} I(x > 0)$, where $\theta > 0$.

   (a) Derive a general expression for the Maximum Likelihood Estimator (MLE). Carry out the second derivative test to make sure you really have a maximum. Circle your final answer.

   (b) Use these data to calculate a numerical estimate: `0.28, 1.72, 0.08, 1.22, 1.86, 0.62, 2.44, 2.48, 2.96` Answer: 1.517778

9. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\theta) = 2e^{-2(x-\theta)} I(x \geq \theta)$.

   (a) Derive a general expression for a Method of Moments Estimator (MOM).

   (b) Derive a general expression for the Maximum Likelihood Estimator (MLE).

   (c) Is the MLE unbiased? Do the calculation and answer Yes or No.

   (d) Is the MLE consistent? Do the calculation and answer Yes or No.

   (e) Calculate both your estimates for the following data: `8.51, 6.11, 6.25, 6.13, 8.43, 6.34, 6.49`.

10. Let $X_1, \ldots, X_n$ be a random sample from Normal(μ, σ^2) distribution, with both parameters unknown. Find the Maximum Likelihood Estimators. Make sure you prove that the likelihood function really has a maximum at that point.

11. Let $X_1, \ldots, X_n$ be a random sample from a Uniform(L, R) distribution. Find the Maximum Likelihood Estimators of L and R.

12. Do Exercise 6.2.11. Delete the final words "in terms of the likelihood function."

13. Do Exercises 6.2.12, 6.2.14 and 6.2.16 in the text.

14. Let $X_1, \ldots, X_n$ be independent random variables, with $X_i \sim \text{Binomial}(m_i, \theta)$. The unknown parameter is θ, while $m_1, \ldots, m_n$ are fixed, observable constants. They are all integers greater than or equal to one.

   (a) Derive the least-squares estimator of θ. Show your work. The final answer is a formula.

   (b) Is the least-squares estimator unbiased? Answer Yes or No and show your work.

   (c) Derive the Maximum Likelihood Estimator of θ. Show your work; don't forget the second derivative test. The final answer is a formula.

   (d) Is the Maximum Likelihood Estimator unbiased? Answer Yes or No and show your work.

   (e) Show that the Maximum Likelihood Estimator is consistent. Use the fact that $m_i \geq 1$ and squeeze.

   (f) Calculate both estimates for the following data:

m	10	13	5	18	17	6	14	5	10	14
x	4	7	0	14	6	4	5	4	3	4

15. For $i = 1, \ldots, n$, let $Y_i = \beta x_i + E_i$, where

   $x_1, \ldots, x_n$ are fixed, observable constants like drug doses.

   $E_1, \ldots, E_n$ are independent random variables with expected value zero and variance σ^2.

   β and σ^2 are unknown constants (parameters).

   (a) What is $E(Y_i)$?

   (b) What is $Var(Y_i)$?

   (c) Derive a formula for the least squares estimate of β.

   (d) Is this least squares estimate unbiased? Answer Yes or No and show your work.

   (e) Assuming that $\lim_{n\to\infty} \frac{1}{\sum_{i=1}^n x_i^2} = 0$, show that $\widehat{\beta}_n$ is consistent.

   (f) Calculate the least squares estimate of β for the following data:

x	1	6	3	7	2
y	6	37	12	52	4

   (g) Now assume that $E_i \sim \text{Normal}(0, \sigma^2)$; this is almost universal in statistics courses, software and applications.

      i. What is the distribution of Y_i? You should be able to just write down the answer.

      ii. Find the MLEs of β and σ^2. Show your work.

      iii. Using the fact that linear combinations of independent normals are normal, what is the distribution of $\widehat{\beta}$, including the parameters?

      iv. Assume that $\sigma^2 = \sigma_0^2$ is fixed and known. Derive a $(1-\alpha)100\%$ confidence interval for β. (In reality, σ^2 is never known and you would use the t distribution, but that's a longer story.)

      v. Assuming $\sigma^2 = 4$, calculate your confidence interval for the data of Question (15f). The answer is two numbers, a lower confidence limit and an upper confidence limit.

16. Consider the model of Question 15, except that now there is an unknown intrcept: $Y_i = \beta_0 + \beta_1 x_i + E_i$. The unknown parameters are β_0, β_1 and σ^2.
    (a) Obtain the least-squares estimates of β_0 and β_1. Show your work. Your answer is two formulas.
    (b) Suppose $E_i \sim N(0, \sigma^2)$. Show that the MLEs of β_0 and β_1 are the same as the least-squares estimates.
