STA 260s20 Assignment Two: Unbiasedness and Consistency

These homework problems are not to be handed in. They are preparation for Quiz 2 (Week of Jan. 20) and Term Test 1. **Please try each question before looking at the solution**.

1. Let $X_1, \ldots X_n$ be independent Binomial random variables with parameters $m = 3$ (known) and θ (unknown); see the formula sheet. Let $\widehat{\Theta}_n = \frac{1}{3}\overline{X}_n$.
   (a) What is the parameter space Ω for this problem?
   (b) Show that $\widehat{\Theta}_n$ is unbiased.
   (c) Show that $\widehat{\Theta}_n$ is consistent.
2. Let $X_1, \ldots X_n$ be a random sample from a distribution with density $f(x|\theta) = \theta x^{\theta-1}\, I(0 < x < 1)$, where the parmeter $\theta > 0$.
   (a) What is the parameter space Ω for this problem?
   (b) Is $\overline{X}_n$ an unbiased estimator of θ? Answer Yes or No and prove your answer.
   (c) Is $\overline{X}_n$ a consistent estimator of θ? Answer Yes or No and prove your answer.
3. Let $X_1, \ldots X_n$ be independent random variables with expected value μ and variance σ^2. Other than that, the distributions of the X_i are unspecified.
   (a) Show that $S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X}_n)^2}{n-1}$ is an unbiased estimator of σ^2.
   (b) Suppose that μ is known. Is $\widehat{\sigma}^2_n = \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2$ a biased estimator of σ^2, or is it unbiased? Show your work.
   (c) Why does the Law of Large Numbers imply that $\widehat{\sigma}^2_n$ is consistent?
   (d) There is one little hole in the argument for consistency. What is it?
4. Let $X_1, \ldots X_n$ be independent Poisson random variables with unknown parameter λ.
   (a) What is the parameter space Ω for this problem?
   (b) Suggest an estimator of λ that is unbiased and consistent.
   (c) Suggest another estimator of λ. Is it also unbiased? How do you know?
   (d) Using the definition of a limit, it may easily be shown that if the sequence of constants $a_n \rightarrow a$ as an ordinary limit as $n \rightarrow \infty$, then $a_n \stackrel{p}{\rightarrow} a$ as a sequence of degenerate random variables. Using this fact and the multivariable version of continuous mapping for convergence in probability, show that S^2 is consistent for λ.
   (e) Finally, here is a silly estimator: $\widehat{\lambda} = (X_1 + X_2)/2$.
      i. Is $\widehat{\lambda}$ unbiased? Why or why not?
      ii. Is $\widehat{\lambda}$ consistent? Why or why not?
      iii. Why is $\widehat{\lambda}$ silly?

5. Let $X_1, \ldots X_n$ be independent Uniform $(0, \theta)$ random variables.

   (a) What is the parameter space Ω for this problem?

   (b) Write the cumulative distribution function $F_{X_i}(x|\theta)$ using indicator functions. Show your work.

   (c) Let $T_n = \max(X_i)$. Find the cumulative distribution function of T_n. Show your work. Write the final answer using indicator functions.

   (d) Find the density function of T_n. Write it using indicator functions.

   (e) Is T_n unbiased for θ? Answer Yes or No and show your work.

   (f) Show that T_n is consistent for θ using the definition.

   (g) Show that T_n is consistent for θ using the variance rule.

   (h) Give an unbiased estimator of θ based on T_n. That is, fix up T_n a bit so it's unbiased. Call the new estimator $\widehat{\Theta}_1$.

   (i) Let $\widehat{\Theta}_2 = 2\overline{X}_n$. Show that $\widehat{\Theta}_2$ is unbiased and consistent.

   (j) In terms of variance, which is preferable, $\widehat{\Theta}_1$ or $\widehat{\Theta}_2$?

6. For $i = 1, \ldots, n$, let $Y_i = \beta x_i + \epsilon_i$, where

   $x_1, \ldots, x_n$ are fixed, known constants

   $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed Normal(0,σ^2) random variables; the parameters β and σ^2 are unknown.

   This is a very simple regression model. For example, the x_i values could be drug doses, and the Y_i could be response to the drug. Naturally, the main interest is in β, because β is the connection between dose and response.

   (a) A suggested estimator is $\widehat{\beta}_1 = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}$.

      i. Is $\widehat{\beta}_1$ unbiased for β? Answer Yes or No and show your work.

      ii. Assume that $\lim_{n\to\infty} \frac{1}{\sum_{i=1}^n x_i^2} = 0$, which is reasonable for drug doses. Is $\widehat{\beta}_1$ consistent for β? Answer Yes or No and show your work.

   (b) Another suggested estimator is $\widehat{\beta}_2 = \frac{\overline{Y}_n}{\overline{x}_n}$.

      i. Is $\widehat{\beta}_2$ unbiased for β? Answer Yes or No and show your work.

      ii. Is $\widehat{\beta}_2$ consistent for β? Answer Yes or No and show your work. Note that you can't use the Law of Large Numbers, because the Y_i don't have the same expected value. However, you may assume that $\lim_{n\to\infty} \overline{x}_n = c \neq 0$, which is reasonable for drug doses.

   (c) It is tough to show, but $Var(\widehat{\beta}_1) \leq Var(\widehat{\beta}_2)$. Do you feel like giving it a try? This will not be on any test or exam.

7. Let $X_1, \ldots X_n$ be independent Exponential (λ) random variables.
   (a) Suggest a reasonable estimator for λ.
   (b) It is easy to see that your estimator is consistent. Why?
   (c) Unbiasedness is another issue. First, derive the distribution of $\overline{X}_n$ and write the density $f_{\overline{X}_n}(\overline{x}|\lambda)$.
   (d) Now directly calculate $E\left(1/\overline{X}_n\right)$. Is this estimator unbiased for λ?
   (e) Show that $\frac{n-1}{\sum_{i=1}^n X_i}$ is unbiased for λ.
   (f) Show that $\frac{n-1}{\sum_{i=1}^n X_i}$ is consistent for λ.
8. Let $X_1, \ldots X_n$ be independent random variables with expected value μ and variance σ^2. Other than that, the distributions of the X_i are unspecified. We seek to estimate μ with the linear combination $L = a_1X_1 + \cdots + a_nX_n = \sum_{i=1}^n a_iX_i$, where $a_1, \ldots, a_n$ are constants.
   (a) What condition on $a_1, \ldots, a_n$ is required for L to be an unbiased estimator of μ? Show your work.
   (b) $\overline{X}_n$ is one such linear combination. What are the coefficients $a_1, \ldots, a_n$?
   (c) Show that the variance of $\overline{X}_n$ is less than the variance of any other unbiased linear combination L. That is, $\overline{X}_n$ is the Best Linear Unbiased Estimator (BLUE).
