STA 260s20 Assignment Ten: More Estimation[1]

The following homework problems are not to be handed in. They are preparation for the final exam. **Please try each question before looking at the solution**. Use the formula sheet.

1. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter θ.

   (a) Give a one-dimensional sufficient statistic for θ. Show your work and circle your answer.

   (b) Calculate your sufficient statistic for the following set of data: 1 0 1 0 0. Your answer is a single number; circle it. My answer is 2, but yours may be different and still correct, if you arrived at another sufficient statistic.

2. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ.

   (a) Give a one-dimensional sufficient statistic for λ. In addition to being sufficient, your answer must also be an unbiased estimator. Show your work and circle your answer. You do not need to prove that your estimator is unbiased.

   (b) Calculate your sufficient statistic for the following set of data: 14 10 8 8. Your answer is a single number; circle it. My answer is 10.

3. Let $X_1, \ldots, X_n$ be a random sample from a Gamma distribution with parameters $\alpha = \theta$ and $\lambda = \frac{1}{2}$.

   (a) Give a one-dimensional sufficient statistic for θ.

   (b) Calculate your sufficient statistic for the following set of data: 0.706 2.154 2.367 4.039 2.155 1.678. Your answer is a single number; circle it. My answer is 52.57288, but yours may be different and still correct, if you arrived at another sufficient statistic.

4. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution with parameters L and R.

   (a) Give a two-dimensional sufficient statistic for (L, R). Show your work and circle your answer.

   (b) Calculate your sufficient statistic for the following set of data: 5.103 6.400 5.415 4.198 4.817 5.907. Your answer is a pair of numbers; circle them. My answer is (4.198, 6.4), but yours may be different and still correct, if you arrived at another sufficient statistic.

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5. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with parameters μ and σ^2.

   (a) Give a two-dimensional sufficient statistic for (μ, σ^2). In addition to being sufficient, your statistics must also be unbiased estimators. Show your work and circle your answer. You do not need to prove that your estimators are unbiased.

   (b) Calculate your sufficient statistic for the following set of data: 100.3 100.6 96.5 99.3 104.1. Your answer is a pair of numbers; circle them. My answer is (100.16, 7.468).

6. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density

$$f(x|\theta, \delta) = \frac{1}{\theta} e^{-\frac{x-\delta}{\theta}} I(x \geq \delta),$$

   where $\theta > 0$ and δ is any real number.

   (a) Give a two-dimensional sufficient statistic for (θ, δ). Show your work and circle your answer.

   (b) Calculate your sufficient statistic for the following set of data: 11.03 10.34 11.26 10.02 10.42 10.58. Your answer is a pair of numbers; circle them. My answer is (63.65, 10.02), but yours may be different and still correct, if you arrived at another sufficient statistic.

7. Let $X_1, \ldots, X_n$ be a random sample from a discrete distribution with probability mass function $p(x|\theta)$. Show that $E(\ell'(\theta, \mathbf{X})) = 0$. Assume that $\frac{\partial}{\partial\theta}$ can be passed through summation signs with respect to x; this is a "regularity condition". Start with $E\left(\frac{\partial}{\partial\theta} \ln p(X_i|\theta)\right)$.

8. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter θ.

   (a) Suggest a reasonable estimator of θ; call it $\widehat{\Theta}_n$. Is $\widehat{\Theta}_n$ an unbiased estimator of θ? Just answer Yes or No.

   (b) What is $Var(\widehat{\Theta}_n)$? Show a little work this time.

   (c) Find the Cramér-Rao lower bound on the variance for this problem.

   (d) Comparing the variance of $\widehat{\Theta}_n$ to the Cramér-Rao lower bound, is $\widehat{\Theta}_n$ a Minimum Variance Unbiased Estimator? Answer Yes or No.

9. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with $\mu = 0$ and unknown variance $\theta > 0$. Consider the Method of Moments estimator $\widehat{\Theta} = \frac{1}{n}\sum_{i=1}^n X_i^2$.
   (a) Is $\widehat{\Theta}_n$ an unbiased estimator? Show your work and answer Yes or No.
   (b) What is $Var(\widehat{\Theta}_n)$? Hint: What is the distribution of $\frac{1}{\theta}\sum_{i=1}^n X_i^2$?
   (c) Find the Cramér-Rao lower bound on the variance for this problem.
   (d) Comparing the variance of $\widehat{\Theta}_n$ to the Cramér-Rao lower bound, is $\widehat{\Theta}_n$ a Minimum Variance Unbiased Estimator?
10. Let $X_1, \ldots, X_n$ be a random sample from a Geometric distribution with parameter θ.
    (a) Derive the maximum likelihood estimate of θ. Show your work. The answer is a formula. Don't bother with the second derivative test.
    (b) A sample of size $n = 100$ yields $\overline{x}_n = 0.85$. Give the MLE and a 95% confidence interval for θ. The MLE is a number, and the confidence interval is a pair of numbers, a lower confidence limit and an upper confidence limit. Use the Central Limit Theorem for MLEs.
11. As in Question 9, suppose we have data from a normal distribution with mean zero and variance θ. A random sample of size $n = 120$ yields $\sum_{i=1}^n x_i^2 = 480.38$. Please obtain a 95% confidence interval for θ in two ways.
    (a) Using the Central Limit Theorem for MLEs. This gives you an *approximate* 95% confidence interval.
    (b) Using the chi-squared distribution. This yields an *exact* confidence interval.

    The two intervals should be fairly close.
12. Since the model of Question 9 is so appealing, consider these two test statistics for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

    (a) $Y = \frac{1}{\theta_0}\sum_{i=1}^n X_i^2$ Reject when $Y \geq \chi^2_{1-\alpha}(n)$

    (b) $Z_n = \frac{\sqrt{n}(\widehat{\Theta}_n - \theta)}{\sqrt{1/I(\theta_0)}}$ Reject when $Z_n \geq z_{1-\alpha}$

    Find the power of each test for $H_0 : \theta \leq 4$ and $n = 120$ when the true value of θ is 4.25. For each test, show your work and use R to obtain the power, a number between zero and one. Include the R command in your answer; it's very short, like `pnorm(something)`.
