

Sample Questions: Moment-generating functions

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1. Let X have a moment-generating function $M_X(t)$ and let a be a constant. Show $M_{aX}(t) = M_X(at)$.

$$\begin{aligned} M_{aX}(t) &\stackrel{\text{def}}{=} E(e^{aXt}) = E(e^{X(at)}) \\ &= M_X(at) \quad \checkmark \end{aligned}$$

Example MGF of $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$

X_1, \dots, X_n have MGF $M_X(t)$

$$\begin{aligned} M_{\bar{X}}(t) &= M_{\frac{1}{n} \sum X_i}(t) = M_{\sum_{i=1}^n X_i}\left(\frac{t}{n}\right) \\ &\stackrel{\text{ind}}{=} \prod_{i=1}^n M_X\left(\frac{t}{n}\right) = \left(M_X\left(\frac{t}{n}\right)\right)^n \end{aligned}$$

2. Let X have a moment-generating function $M_X(t)$ and let a be a constant. Show $M_{a+X}(t) = e^{at}M_X(t)$.

$$\begin{aligned}M_{a+X}(t) &= E(e^{(a+X)t}) \\&= E(e^{at+Xt}) = E(e^{at} \cdot e^{Xt}) \\&= e^{at} E(e^{Xt}) = e^{at} M_X(t) \quad \checkmark\end{aligned}$$

3. Let X and Y be independent, (continuous) random variables.
 Show $M_{X+Y}(t) = M_X(t) M_Y(t)$. *Be very clear about where you use independence.*

$$M_{X+Y}(t) = E(e^{(X+Y)t})$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x+y)t} f_{X,Y}(x,y) dx dy$$

$$\stackrel{\text{ind}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{xt+yt} f_X(x) f_Y(y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{e^{xt}}_{f_X(x)} \underbrace{e^{yt}}_{f_Y(y)} f_X(x) f_Y(y) dx dy$$

$$= \int_{-\infty}^{\infty} e^{yt} f_Y(y) \underbrace{\int_{-\infty}^{\infty} e^{xt} f_X(x) dx}_{M_X(t)} dy$$

$$M_X(t) \int_{-\infty}^{\infty} e^{yt} f_Y(y) dy = M_X(t) M_Y(t)$$

□

4. Let $Z \sim N(0, 1)$. Calculate $M_Z(t) = E(e^{zt})$

$$= \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \text{Exp} \left(-\frac{1}{2}(z^2 - 2zt + t^2 - t^2) \right) dz$$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz$$

comparo

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$= e^{\frac{1}{2}t^2}$$

5. Let $X \sim N(\mu, \sigma^2)$. Calculate $M_X(t)$.

$$\text{Let } Z = \frac{X - \mu}{\sigma} \sim N(0, 1) \quad M_Z(t) = e^{t^2/2}$$

$$X = \sigma Z + \mu \quad M_X(t) = E(e^{(\sigma Z + \mu)t})$$

$$= E(e^{Z(\sigma t)} \cdot e^{\mu t})$$

$$= e^{\mu t} M_Z(\sigma t) = e^{\mu t} e^{\frac{1}{2}(\sigma t)^2}$$

$$= e^{\mu t + \frac{1}{2} \sigma^2 t^2}$$

6. Let $X \sim N(\mu, \sigma^2)$. Show $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$ using moment-generating functions.

$$M_X(t) = e^{\mu t + \frac{1}{2} \sigma^2 t^2}$$

$$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t) = E\left(e^{\left(\frac{X-\mu}{\sigma}\right)t}\right)$$

$$= E\left(e^{X\left(\frac{t}{\sigma}\right) - \frac{\mu t}{\sigma}}\right) = e^{-\frac{\mu t}{\sigma}} E\left(e^{X\left(\frac{t}{\sigma}\right)}\right)$$

$$= e^{-\frac{\mu t}{\sigma}} M_X\left(\frac{t}{\sigma}\right)$$

$$= e^{-\frac{\mu t}{\sigma}} e^{\mu \frac{t}{\sigma} + \frac{1}{2} \sigma^2 \frac{t^2}{\sigma^2}}$$

$$= e^{-\frac{\mu t}{\sigma} + \frac{\mu t}{\sigma} + \frac{1}{2} t^2} = e^{\frac{1}{2} t^2}$$

compare $e^{\mu t + \frac{1}{2} \sigma^2 t^2}$

$$\parallel$$

$$e^{0t + \frac{1}{2} 1^2 t^2}$$

MGF of $N_0(0, 1)$

7. Let $X \sim N(\mu, \sigma^2)$. Find the distribution of $Y = a + bX$ using moment-generating functions.

$$\begin{aligned}M_Y(t) &= M_{a+bX}(t) = E\left(e^{(a+bX)t}\right) \\&= e^{at} E\left(e^{X(bt)}\right) \\&= e^{at} M_X(bt) \left(M_X(t) = e^{at + \frac{1}{2}\sigma^2 t^2} \right) \\&= e^{at} e^{abt + \frac{1}{2}\sigma^2 b^2 t^2} \\&= e^{(a+bt)t + \frac{1}{2}(b^2\sigma^2)t^2}\end{aligned}$$

MGF of $N(a+b\mu, b^2\sigma^2)$

8. Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be independent. Find the distribution of $Y = aX_1 + bX_2 + c$.

$$\begin{aligned}
 M_Y(t) &= E(e^{Yt}) = E(e^{(aX_1 + bX_2 + c)t}) \\
 &= E(e^{ax_1t} e^{bx_2t} e^{ct}) \\
 &\stackrel{\text{ind}}{=} e^{ct} E(e^{X_1(at)}) E(e^{X_2(bt)}) \\
 &= e^{ct} M_{X_1}(at) M_{X_2}(bt) \\
 &= e^{ct} e^{\mu_1 at + \frac{1}{2} \sigma_1^2 a^2 t^2} e^{\mu_2 bt + \frac{1}{2} \sigma_2^2 b^2 t^2} \\
 &= e^{\underbrace{(a\mu_1 + b\mu_2 + c)t}_{\mu'}} + \frac{1}{2} \underbrace{(a^2 \sigma_1^2 + b^2 \sigma_2^2)}_{\sigma'^2} t^2
 \end{aligned}$$

$MGF \text{ of } N(a\mu_1 + b\mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$

9. Let $Z \sim N(0, 1)$ and let $Y = Z^2$. Find the distribution of Y . Note that the MGF of a chi-squared random variable is $M(t) = (1 - 2t)^{-\frac{\nu}{2}}$.

$$M_{Z^2}(t) = E(e^{Z^2 t})$$

$$= \int_{-\infty}^{\infty} e^{z^2 t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - z^2 2t)} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-2t)z^2} dz \quad \text{for } t < \frac{1}{2} \text{ converges}$$

$$= (1-2t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{(1-2t)^{-\frac{1}{2}} \sqrt{2\pi}} e^{-\frac{1}{2(1-2t)}z^2} dz$$

$$(1-2t)^{-\frac{1}{2}}$$

$$(1-2t)^{-\frac{\nu}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} N(0, \sigma^2 = (1-2t)^{-1})$$

MGF of $\chi^2(\nu=1)$

$$M_X(t) = (1 - 2t)^{-\nu/2}$$

10. Independently for $i = 1, \dots, n$, let $Y_i \sim \chi^2(\nu_i)$. Find the distribution of $W = \sum_{i=1}^n Y_i$.

$$\begin{aligned} M_W(t) &= M_{\sum_{i=1}^n Y_i}(t) \stackrel{\text{ind}}{=} \prod_{i=1}^n M_{Y_i}(t) \\ &= \prod_{i=1}^n (1 - 2t)^{-\nu_i/2} = (1 - 2t)^{-\sum_{i=1}^n \nu_i/2} \end{aligned}$$

MGF of $\chi^2\left(\sum_{i=1}^n \nu_i\right)$

11. Independently for $i = 1, \dots, n$, let $X_i \sim N(\mu_i, \sigma_i)$. What is the distribution of $Y = \sum_{i=1}^n \left(\frac{X_i - \mu_i}{\sigma_i} \right)^2$? Justify your answer.

$Y \sim \chi^2(n)$ because

$$Z = \frac{X_i - \mu_i}{\sigma_i} \sim N(0, 1) \text{ by Prob 5}$$

$$Z^2 \sim \chi^2(1) \text{ by Prob 9}$$

Sum of ind χ^2 is $\chi^2(\sum \nu_i)$ by Prob 10

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<http://www.utstat.toronto.edu/~brunner/oldclass/256f19>