Sample Questions: Moment-generating functions

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1. Let X have a moment-generating function $M_X(t)$ and let a be a constant. Show $M_{aX}(t) = M_X(at)$.

$$M_{aX}(t) \stackrel{\text{def}}{=} E(e^{aXt}) = E(e^{X(at)})$$

$$= M_X(at)$$

Example MGF of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$X_1 \ldots X_n$ have MGF $M_X(t)$

$$M_{\bar{X}}(t) = M_{\frac{1}{n}\sum X_i}(t) = M_{\sum_{i=1}^{n} X_i}\left(\frac{t}{n}\right)$$

$$\stackrel{\text{ind}}{=} \prod_{i=1}^{n} M_X\left(\frac{t}{n}\right) = \left(M_X\left(\frac{t}{n}\right)\right)^n$$

2. Let X have a moment-generating function $M_X(t)$ and let a be a constant. Show $M_{a+X}(t) = e^{at} M_X(t)$.

$$M_{a+X}(t) = E\left(e^{(a+X)t}\right)$$

$$= E\left(e^{at+Xt}\right) = E\left(e^{at} \cdot e^{Xt}\right)$$

$$= e^{at} E\left(e^{Xt}\right) = e^{at} M_X(t)$$

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3. Let X and Y be independent, (continuous) random variables. Show $M_{X+Y}(t) = M_X(t)\,M_Y(t)$. Be very clear about where you use independence.

$$M_{X+Y}(t) = E\left(e^{(X+Y)t}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(x+y)t} f_{X,Y}(x,y)\,dx\,dy$$

$$\overset{ind}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{xt+yt} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{xt} e^{yt} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} e^{yt} f_Y(y) \underbrace{\int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx}_{M_X(t)}\,dy$$

$$M_X(t)\int_{-\infty}^{\infty} e^{yt} f_Y(y)\,dy = M_X(t)\,M_Y(t)$$

$\boxtimes$

4. Let $Z \sim N(0,1)$. Calculate $M_Z(t)$. $= E(e^{zt})$

$$= \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \operatorname{Exp} -\frac{1}{2}\left(z^2 - 2zt + t^2 - t^2\right) dz$$

$$= e^{\frac{1}{2}t^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2}}_{\text{compare } \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}} \, dz$$

$$= e^{\frac{1}{2}t^2} \cdot 1$$

5. Let $X \sim N(\mu, \sigma^2)$. Calculate $M_X(t)$.

Let $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ $M_Z(t) = e^{t^2/2}$

$X = \sigma Z + \mu$ $M_X(t) = E(e^{(\sigma Z + \mu)t})$

$= E\left(e^{Z(\sigma t)} \cdot e^{\mu t}\right)$

$= e^{\mu t} M_Z(\sigma t) = e^{\mu t} e^{\frac{1}{2}(\sigma t)^2}$

$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

6. Let $X \sim N(\mu, \sigma^2)$. Show $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ using moment-generating functions.

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t) = E\left(e^{\left(\frac{X-\mu}{\sigma}\right)t}\right)$$

$$= E\left(e^{X\left(\frac{t}{\sigma}\right) - \frac{\mu t}{\sigma}}\right) = e^{-\frac{\mu t}{\sigma}} E\left(e^{X\left(\frac{t}{\sigma}\right)}\right)$$

$$= e^{-\frac{\mu t}{\sigma}} M_X\left(\frac{t}{\sigma}\right)$$

$$= e^{-\frac{\mu t}{\sigma}} e^{\mu \frac{t}{\sigma} + \frac{1}{2}\cancel{\sigma^2}\frac{t^2}{\cancel{\sigma^2}}}$$

$$= e^{-\cancel{\frac{\mu t}{\sigma}} + \cancel{\frac{\mu t}{\sigma}} + \frac{1}{2}t^2} = e^{\frac{1}{2}t^2}$$

compare $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$$e^{\frac{1}{2}t^2} = e^{0t + \frac{1}{2}1^2 t^2}$$

MGF of $N(0,1)$

7. Let $X \sim N(\mu, \sigma^2)$. Find the distribution of $Y = a + bX$ using moment-generating functions.

$$M_Y(t) = M_{a+bX}(t) = E\left(e^{(a+bX)t}\right)$$

$$= e^{at} E\left(e^{X(bt)}\right)$$

$$= e^{at} M_X(bt) \quad \left(M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)$$

$$= e^{at} e^{\mu b t + \frac{1}{2}\sigma^2 b^2 t^2}$$

$$= e^{\overset{\mu'}{(a+b\mu)}t + \frac{1}{2}\overset{\sigma^{2\prime}}{(b^2\sigma^2)}t^2}$$

MGF of $N(a+b\mu, b^2\sigma^2)$

8. Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be independent. Find the distribution of $Y = aX_1 + bX_2 + c$.

$$M_Y(t) = E(e^{Yt}) = E\left(e^{(aX_1+bX_2+c)t}\right)$$

$$= E\left(e^{aX_1t} e^{bX_2t} e^{ct}\right)$$

$$\overset{\text{ind}}{=} e^{ct} E\left(e^{X_1(at)}\right) E\left(e^{X_2(bt)}\right)$$

$$= e^{ct} M_{X_1}(at) M_{X_2}(bt)$$

$$= e^{ct} e^{\mu_1 ta + \frac{1}{2}\sigma_1^2 a^2 t^2} e^{\mu_2 bt + \frac{1}{2}\sigma_2^2 b^2 t^2}$$

$$= e^{\underbrace{(a\mu_1+b\mu_2+c)}_{\mu'}t + \frac{1}{2}\underbrace{(a^2\sigma_1^2+b^2\sigma_2^2)}_{\sigma^{2\prime}}t^2}$$

MGF of $N(a\mu_1+b\mu_2+c,\ a^2\sigma_1^2+b^2\sigma_2^2)$

9. Let $Z \sim N(0,1)$ and let $Y = Z^2$. Find the distribution of Y. Note that the MGF of a chi-squared random variable is $M(t) = (1-2t)^{-\frac{\nu}{2}}$.

$$M_{z^2}(t) = E(e^{z^2 t})$$

$$= \int_{-\infty}^{\infty} e^{z^2 t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - z^2 2t)} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-2t)z^2} \, dz \quad \text{for } t < \tfrac{1}{2} \text{ converges}$$

$$= (1-2t)^{-\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{(1-2t)^{-1/2}\sqrt{2\pi}} e^{-\frac{1}{2(1-2t)^{-1}} z^2} \, dz}_{=1 \quad N(0,\, \sigma^2 = (1-2t)^{-1})}$$

$$(1-2t)^{-\frac{1}{2}}$$

$$(1-2t)^{-\nu/2}$$

MGF of $\chi^2(\nu = 1)$

$$M_X(t) = (1-2t)^{-\nu/2}$$

10. Independently for $i = 1, \ldots, n$, let $Y_i \sim \chi^2(\nu_i)$. Find the distribution of $W = \sum_{i=1}^n Y_i$.

$$M_W(t) = M_{\sum_{i=1}^n Y_i}(t) \overset{\text{ind}}{=} \prod_{i=1}^n M_{Y_i}(t)$$

$$= \prod_{i=1}^n (1-2t)^{-\nu_i/2} = (1-2t)^{-\sum_{i=1}^n \nu_i/2}$$

MGF of $\chi^2\left(\sum_{i=1}^n \nu_i\right)$

11. Independently for $i = 1, \ldots, n$, let $X_i \sim N(\mu_i, \sigma_i)$. What is the distribution of $Y = \sum_{i=1}^n \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$? Justify your answer.

$Y \sim \chi^2(n)$ because

$Z = \frac{X_i - \mu_i}{\sigma_i} \sim N(0,1)$ by prob 5

$Z^2 \sim \chi^2(1)$ by prob 9

sum of ind χ^2 is $\chi^2(\Sigma \nu_i)$ by Prob 10