Sample Questions: Transformations

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1. Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent. Using the convolution formula, find the probability mass function of $Z = X + Y$ and identify it by name.

Use $P_Z(z) = \sum_x P_X(x)\, P_Y(z-x)$ $\quad x \le z$

Poisson pmf $\frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0, 1, \ldots$

For $z = 0, 1, \ldots$

$$P_Z(z) = \sum_{x=0}^{z} \frac{e^{-\lambda_1}\lambda_1^x}{x!} \frac{e^{-\lambda_2}\lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \sum_{x=0}^{z} \frac{z!}{x!\,(z-x)!} \lambda_1^x \lambda_2^{z-x}$$

Think of $(a+b)^n = \sum_{k=0}^{n} a^k b^{n-k}$

$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$= \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^z}{z!}$$

So

$$f_Z(z) = \begin{cases} \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^z}{z!} & \text{for } z = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

POISSON$(\lambda_1 + \lambda_2)$

2. Independently for $i = 1, \ldots, n$, let $X_i \sim \text{Poisson}(\lambda_i)$, and let $Y_n = \sum_{i=1}^{n} X_i$. Using the last problem, what is the probability distribution of Y_n?

$$Y_n \sim \text{Poisson}\left(\sum_{i=1}^{n} \lambda_i\right)$$

Show ↑. True for $n=2$ by Prob 2

Induction hypothesis: $Y_{n-1} \sim P\left(\sum_{i=1}^{n-1} \lambda_i\right)$

Then $Y_n = Y_{n-1} + X_n \sim \text{Poisson}\left(\sum_{i=1}^{n-1} \lambda_i + \lambda_n\right)$

Prob 2

3 without replacement

X = # of Red Balls Selected

$$P(X=x) = \frac{\binom{n_1}{x}\binom{n_2}{3-x}}{\binom{n_1+n_2}{3}}$$

USE $P_Z(z) = \sum_{x \in A} P_X(x) P_Y(z-x)$

3. Let $X \sim \text{Binomial}(n_1, \theta)$ and $Y \sim \text{Binomial}(n_2, \theta)$ be independent. Using the convolution formula, find the probability mass function of $Z = X + Y$ and identify it by name.

$$A = \{x : x \text{ is an integer}, x \geq 0, x \leq n_1, z - x \geq 0, z - x \leq n_2\}$$

$P(Z = z) \geq 0$ for $z = 0, 1, \ldots, n_1 + n_2$

$$P_Z(z) = \sum_{x \in A} \binom{n_1}{x} \theta^{x} (1-\theta)^{n_1 - x} \binom{n_2}{z-x} \theta^{z-x} (1-\theta)^{n_2 - z + x}$$

$$= \sum_{x \in A} \theta^{\cancel{x} + z \cancel{- x}} (1-\theta)^{n_1 \cancel{- x} + n_2 - z \cancel{+ x}} \binom{n_1}{x}\binom{n_2}{z-x}$$

$$= \binom{n_1+n_2}{z} \theta^{z} (1-\theta)^{n_1+n_2-z} \underbrace{\sum_{x \in A} \frac{\binom{n_1}{x}\binom{n_2}{z-x}}{\binom{n_1+n_2}{z}}}_{=1}$$

BINOMIAL

HYPERGEOMETRIC

So $Z \sim \text{Binomial}(n_1 + n_2, \theta)$

4. Let $X_1, \dots, X_n$ be independent Bernoulli random variables with parameter θ, and let $Y_n = \sum_{i=1}^{n} X_i$. Using the last problem, what is the probability distribution of Y?

Because $X_i \sim \text{Binomial}(1, \theta)$

$Y_n \sim \text{Binomial}(n, \theta)$

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etc.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

5. Let X and Y be independent exponential random variables with parameter λ. Using the convolution formula, find the probability density function of $Z = X + Y$ and identify it by name.

$f_Y(z-x) \geq 0$ iff $z - x \geq 0 \Leftrightarrow x \leq z$

$$f_Z(z) = \int_0^z \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)}\, dx$$

$$= \lambda^2 \int_0^z e^{-\lambda(\cancel{x} + z - \cancel{x})}\, dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z 1\, dx = \lambda^2 e^{-\lambda z} z \text{ for } z \geq 0$$

$$f_Z(z) = \begin{cases} \dfrac{\lambda^2}{\Gamma(2)} e^{-\lambda z} z^{2-1} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

GAMMA ($\alpha = 2$, $\lambda = \lambda$)

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1(y_1,y_2), x_2(y_1,y_2)) \cdot \text{abs}\,|J|$$

6. Let X_1 and X_2 be independent standard normal random variables. Find the probability density function of $Y_1 = X_1/X_2$.

$$Y_1 = \frac{X_1}{X_2} \Rightarrow X_1 = Y_1 Y_2$$

$$Y_2 = X_2 \qquad X_2 = Y_2$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} = y_2 & \frac{\partial x_1}{\partial y_2} = y_1 \\ \frac{\partial x_2}{\partial y_1} = 0 & \frac{\partial x_2}{\partial y_2} = 1 \end{vmatrix}$$

$$ad - bc = y_2$$

So $f_{Y_1 Y_2}(y_1, y_2)$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_1 y_2)^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_2^2} \cdot |y_2|$$

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(1+y_1^2)y_2^2} \, |y_2| \, dy_2$$

$$\overset{\text{SYMMETRY}}{=} \frac{1}{\cancel{2}\pi (1+y_1^2)} \cancel{2} \int_0^\infty e^{-\frac{1}{2}(1+y_1^2)y_2^2} (1+y_1^2) y_2 \, dy_2$$

$$u = \frac{1}{2}(1+y_1^2) y_2^2$$

$$du = \cancel{\tfrac{1}{2}}(1+y_1^2)\cancel{2} y_2 \, dy_2$$

y_2	u
∞	∞
0	0

$$= \frac{1}{\pi} \frac{1}{1+y_1^2} \overbrace{\int_0^\infty e^{-u} \, du}^{=1}$$

Use $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1(y_1,y_2), x_2(y_1,y_2)) \cdot \text{abs}|J|$ to show

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

7. Use the Jacobian method to prove the convolution formula for continuous random variables.

$$Y_1 = X_1 + X_2 \Rightarrow X_1 = Y_1 - Y_2$$
$$Y_2 = X_2 \qquad X_2 = Y_2$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} = 1 & \frac{\partial x_1}{\partial y_2} = -1 \\ \frac{\partial x_2}{\partial y_1} = 0 & \frac{\partial x_2}{\partial y_2} \; 1 \end{vmatrix} = 1$$

So

$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1}(y_1 - y_2)\, f_{X_2}(y_2) \quad \text{AND}$$

$$\underset{\substack{\uparrow\\ Z}}{f_{Y_1}}(\underset{\substack{\uparrow\\ z}}{y_1}) = \int_{-\infty}^{\infty} \underset{\substack{\uparrow\\ X}}{f_{X_2}}(\underset{\substack{\uparrow\\ x}}{y_2})\, \underset{\substack{\uparrow\\ Y}}{f_{X_1}}(\underset{\substack{\uparrow\\ z}}{y_1} - \underset{\substack{\uparrow\\ x}}{y_2})\, \underset{\substack{\uparrow\\ dx}}{dy_2}$$

8. Prove $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. $\quad \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt$

$t = \frac{1}{2}x^2 \qquad dt = \cancel{\tfrac{1}{2}}\,\cancel{2}\,x\,dx$

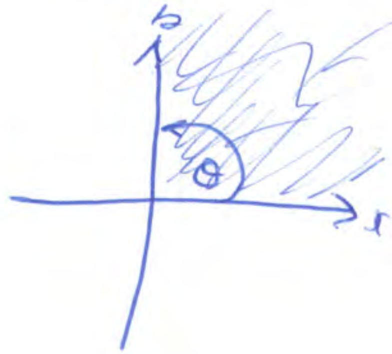
$$\begin{array}{c|c} t & x \\ \hline \infty & \infty \\ 0 & 0 \end{array}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-\frac{1}{2}x^2}\left(\tfrac{1}{2}x^2\right)^{-\frac{1}{2}} x\,dx$$

$$= \int_0^\infty e^{-\frac{1}{2}x^2}\left(\frac{2}{\cancel{x^2}}\right)^{\frac{1}{2}} \cancel{x}\,dx = \sqrt{2}\int_0^\infty e^{-\frac{1}{2}x^2}\,dx$$

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \int_0^\infty \sqrt{2}\, e^{-\frac{1}{2}x^2}\,dx \int_0^\infty \sqrt{2}\, e^{-\frac{1}{2}y^2}\,dy$$

$$= 2\int_0^\infty\int_0^\infty e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$$



$$= 2\int_0^{\pi/2}\int_0^\infty e^{-\frac{1}{2}r^2}\; r\,dr\,d\theta$$

$u = \frac{1}{2}r^2$

$du = \cancel{\tfrac{1}{2}}\,\cancel{2}\,r\,dr$

$$\begin{array}{c|c} r & u \\ \hline \infty & \infty \\ 0 & 0 \end{array}$$

$$= 2\int_0^{\pi/2} \underbrace{\int_0^\infty e^{-u}\,du}_{=1}\, d\theta \quad = 2\int_0^{\pi/2} d\theta = 2\left(\tfrac{\pi}{2} - 0\right)$$

$$= \pi$$

$$\Longrightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad \blacksquare$$

standard use $dx\,dy = r\,dr\,d\theta$

9. Show that the standard normal probability density function integrates to one.

$$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad I = \int_{-\infty}^{\infty} f_z(z)\,dz \overset{\text{show}}{=} 1$$

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$$

$x^2+y^2=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}r^2}\, r\,dr\,d\theta$$

$u = \frac{1}{2}r^2 \quad du = \frac{1}{\cancel{2}}\cancel{2}r\,dr$

r	u
∞	∞
0	0

$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\int_0^{\infty} e^{-u}\,du}_{=1}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 1\,d\theta = \frac{1}{2\pi}(2\pi - 0) = 1$$

$$= I^2 \Longrightarrow I = 1$$

We'll see

$$f_{Y_1}(y_1) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y_1^{a-1}(1-y_1)^{b-1} \text{ for } 0 \le y_1 \le 1$$
$$0 \text{ otherwise}$$

10. The random variables X_1 and X_2 are independent. X_1 has a gamma distribution with parameters $\alpha = a$ and $\lambda = 1$, and X_2 has a gamma distribution with parameters $\alpha = b$ and $\lambda = 1$. Let $Y_1 = \frac{X_1}{X_1+X_2}$ and $Y_2 = X_1 + X_2$.

   (a) Give the joint density of Y_1 and Y_2. Factor, separating y_1 and y_2 as much as possible. In your final statement of the answer to this part, *specify where the joint density is non-zero.*

$$y_1 = \frac{x_1}{x_1+x_2} \Rightarrow x_1 = y_1 y_2$$

$$y_2 = x_1 + x_2 \quad x_2 = y_2 - x_1 = y_2 - y_1 y_2 = y_2(1-y_1)$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} = y_2 & \frac{\partial x_1}{\partial y_2} y_1 \\ \frac{\partial x_2}{\partial y_1} = -y_2 & \frac{\partial x_2}{\partial y_2} = 1-y_1 \end{vmatrix}$$

$$= y_2(1-y_1) - -y_1 y_2 = y_2 - \cancel{y_1 y_2} + \cancel{y_1 y_2} = y_2$$

Now for $0 \le y_1 \le 1$ and $y_2 \ge 0$,

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1}(y_1 y_2) \cdot f_{X_2}(y_2(1-y_1)) \cdot y_2$$

$$= \frac{1^a}{\Gamma(a)} e^{-y_1 y_2}(y_1 y_2)^{a-1} \cdot \frac{1}{\Gamma(b)} e^{-y_2(1-y_1)} (y_2(1-y_1))^{b-1} y_2$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} e^{-(\cancel{y_1 y_2} + y_2 - \cancel{y_1 y_2})} y_1^{a-1}(1-y_1)^{b-1} \cdot y_2^{a-1+b\cancel{-1+1}}$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} y_1^{a-1}(1-y_1)^{b-1} e^{-y_2} y_2^{a+b-1}$$

so

b) $$f_{Y_1}(y_1) = \frac{1}{\Gamma(a)\,\Gamma(b)}\, y_1^{a-1}(1-y_1)^{b-1} \cdot \underbrace{\int_0^\infty e^{-y_2}\, y_2^{a+b-1}\, dy_2}_{\Gamma(a+b)}$$

$$= \begin{cases} \dfrac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\, y_1^{a-1}(1-y_1)^{b-1} & \text{for } 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Beta(a, b)

$0 \le y_1 \le 1$

$y_2 \ge 0$