Sample Questions: Limits

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1. Let X_n have an exponential distribution with parameter $\lambda = n$. Pick one of these and prove your answer.
   - $X_n \stackrel{p}{\rightarrow} 0$
   - $X_n \stackrel{p}{\rightarrow} 1$
   - X_n does not converge in probability to a constant.

2. Let X_n have an exponential distribution with parameter $\lambda = n$, and let $Y_n = \frac{5X_n+2}{X_n+1}$. To what target does Y_n converge in probability?

3. Let $X_n \sim$ Uniform(0,n). Does X_n converge in probability to a constant? Answer Yes or No and prove your answer.

4. Let $X_n \sim$ Uniform$(0,n)$ and let $Y_n = \frac{X_n}{X_n+1}$. Prove $Y_n \xrightarrow{p} 1$.

5. Let X be a random variable with expected value μ and variance σ^2, and let $Y_n = \frac{X}{n}$.

(a) Show $Y_n \xrightarrow{p} 0$.

(b) What if X is Cauchy? Does the result still hold?

6. Let X_n have a Poisson distribution with parameter $n\lambda$, where $\lambda > 0$. This means $E(X_n) = Var(X_n) = n\lambda$. Let $Y_n = \frac{X_n}{n}$.

(a) For what values of y is $P(Y_n = y) > 0$?

(b) Does Y_n converge in probability to a constant? Answer Yes or No and "prove" your answer.

7. Let the discrete random variable X_n have probability mass function

$$p_X(x) = \begin{cases} \frac{1}{n} & \text{for } x = -n \\ \frac{n-2}{n} & \text{for } x = 0 \\ \frac{1}{n} & \text{for } x = n \end{cases}$$

(a) Try using the variance rule.

(b) Prove $X_n \xrightarrow{p} 0$ anyway.

8. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables from an exponential distribution with parameter λ, so that $E(X_i) = 1/\lambda$ and $Var(X_i) = 1/\lambda^2$. Let $T_n = \frac{n}{\sum_{i=1}^{n} X_i}$. Show $T_n \xrightarrow{p} \lambda$.

9. Let the pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ be selected independently from a joint distribution with $E(X_i) = \mu_x$, $E(Y_i) = \mu_y$, $Var(X_i) = \sigma_x^2$, $Var(Y_i) = \sigma_y^2$, and $Cov(X_i, Y_i) = \sigma_{xy}$. Independence means that X_i and Y_i are independent of X_j and Y_j for $i \neq j$.

   Let $Z_i = aX_i + bY_i$. You would expect $\overline{Z}_n \xrightarrow{p} a\mu_x + b\mu_y$. Is this true? Answer Yes or No and say why.

10. Let $f_{X_n}(x) = \begin{cases} \frac{n+1}{n} x^{1/n} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

We have $X_n \xrightarrow{d} X$. Find the distribution of the target random variable X from the definition.

11. Let $p_{X_n}(x) = \begin{cases} \frac{n+3}{2(n+1)} & \text{for } x = 0 \\ \frac{n-1}{2(n+1)} & \text{for } x = 1 \end{cases}$

Show that $X_n \xrightarrow{d} X \sim \text{Bernoulli}(\theta = \frac{1}{2})$.

12. Let $X_n \sim \text{Normal}(\mu, n)$. Does X_n converge in distribution to a random variable? Answer Yes or No and show your work.

13. Let X be a degenerate random variable with $P(X = c) = 1$.

(a) Sketch $F_X(x)$.

(b) Find the moment-generating function of X.

14. Let X_n be an exponential random variable with parameter $\lambda = n$. It seems that $X_n \xrightarrow{d} X$. Find the distribution of the target random variable X

(a) From the definition.

(b) Using moment-generating functions.

15. Use moment-generating functions to prove the Law of Large Numbers.

16. Let S be the sum of 16 independent Uniform(0,1) random variables. Find the approximate $P(S > 12)$. You may use the fact that a Uniform(0,1) has expected value $\frac{1}{2}$ and variance $\frac{1}{12}$.

17. A multiple choice test has 50 questions with answers ABCD. If a student answers completely at random, what are the chances of getting 15 or more correct? You may use the fact that a Bernoulli(θ) has expected value θ and variance $\theta(1-\theta)$.

18. In a walk-in medical clinic, the time a doctor spends per patient (including paperwork) comes from an unfamiliar skewed distribution with mean 5.1 and standard deviation 4.8 minutes. Find the maximum number of patients that should be scheduled so that the probability of working more than a 7 hour day will be less than 5%.
