Sample Questions: Joint Distributions Part One

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1. The discrete random variables x and y have joint probability mass function $p_{X,Y} = cxy$ for $x = 1, 2, 3$, $y = 1, 2$, and zero otherwise.

   (a) Find the value of the constant c and calculate the marginal ~~frequency~~ probability functions.

	$x=1$	$x=2$	$x=3$	
$y=2$	$2/18$	$4/18$	$6/18$	$\frac{12}{18}$
$y=1$	$1/18$	$2/18$	$3/18$	$\frac{6}{18}$
	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{9}{18}$	

$$c = \frac{1}{18}$$

$$P_X(x) = \begin{cases} 3/18 & \text{for } x=1 \\ 6/18 & \text{for } x=2 \\ 9/18 & \text{for } x=3 \\ 0 & \text{otherwise} \end{cases} \qquad P_Y(y) = \begin{cases} 6/18 & y=1 \\ 12/18 & y=2 \\ 0 & \text{otherwise} \end{cases}$$

   (b) What is $F_X(x)$? $= P(X \le x)$

$$F_X(x) = \begin{cases} 0 & \text{for } x<1 \\ 1/6 & \text{for } 1 \le x < 2 \\ 1/2 & \text{for } 2 \le x < 3 \\ 1 & \text{for } x \ge 3 \end{cases}$$

$y = 4$

3

2. The discrete random variables x and y have joint distribution

4, 4

	$x=1$	$x=2$	$x=3$
$y=2$	3/12	1/12	3/12
$y=1$	1/12	3/12	1/12

-1

Give the following. The answers are numbers. $F_{XY}(x, y) = P(X \le x, Y \le y)$

(a) $F_{X,Y}(1,1) = P(X=1, Y=1) = 1/12$

$F_{X,Y}(2,2) = \frac{8}{12}$

(b) $F_{X,Y}(1.5,4) = 4/12$

$F_{X,Y}(-1,3) = 0$

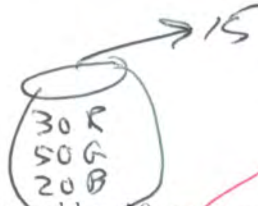
(c) $F_{X,Y}(4,4) = 1$

$F_{X,Y}(6,1.82) = 5/12$

(d) $F_{X,Y}(4,19) = 1$

$F_{X,Y}(0,0) = 0$

	$x=1$	$x=2$	$x=3$
$y=2$	3/12	1/12	3/12
$y=1$	1/12	3/12	1/12



$$\begin{pmatrix} & 15 & \\ x & y & 15-x-y \end{pmatrix}$$

3. A jar contains 30 red marbles, 50 green marbles and 20 blue marbles. A sample of 15 marbles is selected *with replacement*. Let X be the number of red marbles and Y be the number of blue marbles. What is the joint probability mass function of X and Y?

$$p(x,y) = \begin{cases} \begin{pmatrix} & 15 & \\ x & y & 15-x-y \end{pmatrix} .3^x \cdot .2^y \cdot .5^{15-x-y} & \text{for } x=0,\dots,15,\ y=0,\dots,15,\ x+y \le 15 \\ 0 & \text{otherwise} \end{cases}$$

Multinomial

4. This time the selection is without replacement. Again, what is the joint probability mass function of X and Y?

$$p(x,y) = \begin{cases} \dfrac{\binom{30}{x}\binom{20}{y}\binom{50}{15-x-y}}{\binom{100}{15}} & \text{for } x=0,\dots,15,\ y=0,\dots,15,\ x+y \le 15 \\ 0 & \text{otherwise} \end{cases}$$

5. Let $f_{X,Y}(x, y) = \begin{cases} c(x+y) & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$

(a) Find the constant c.

$$1 = c\int_0^1 \int_0^x (x+y)\,dy\,dx$$

$$= c\int_0^1 \int_0^x x\,dy\,dx + c\int_0^1\int_0^x y\,dy\,dx$$

$$= c\int_0^1 x \int_0^x 1\,dy\,dx + c\int_0^1 \frac{y^2}{2}\Big|_0^x dx = c\int_0^1 x\, y\Big|_0^x dx + \frac{c}{2}\int_0^1 x^2\,dx$$

$$= c\int_0^1 x^2\,dx + \frac{c}{2}\int_0^1 x^2\,dx = c\,\frac{x^3}{3}\Big|_0^1 + \frac{c}{2}\,\frac{x^3}{3}\Big|_0^1$$

$$= c\left(\frac{1}{3} + \frac{1}{6}\right) = c\left(\frac{2}{6} + \frac{1}{6}\right) = \frac{c}{2} = 1 \Rightarrow$$

$$c = 2$$

(b) What is $f_X(x)$?

$$0 \le x \le 1,\quad f_X(x) = \int_0^x 2(x+y)\,dy = 2\left(x\int_0^x dy + \int_0^x y\,dy\right)$$

$$= 2\left(x^2 + \frac{y^2}{2}\Big|_0^x\right) = \cancel{2}\left(\frac{2x^2}{\cancel{2}} + \frac{x^2}{\cancel{2}}\right) = 3x^2 \quad \text{so}$$

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

5.5

$$f_{XY}(x,y) = \begin{cases} \dfrac{6}{(x+1)^3(y+1)^4} & \text{for } \begin{matrix} x \ge 0 \\ y \ge 0 \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

Find $F_{XY}(x,y) = P(X \le x, Y \le y)$ For $x \ge 0$ & $y \ge 0$

$$\int_0^y \int_0^x \frac{6}{(s+1)^3(t+1)^4}\, ds\, dt = \int_0^y \frac{6}{(t+1)^4}\left(\int_0^x \frac{1}{(s+1)^3}\, ds\right) dt$$

$$= 6\int_0^y \frac{1}{(t+1)^4} \int_1^{x+1} u^{-3}\, du\, dt \qquad \begin{matrix} u = s+1 \\ du = ds \end{matrix} \qquad \begin{array}{c|c} s & u=s+1 \\ \hline x & x+1 \\ 0 & 1 \end{array}$$

$$= 6\int_0^y \frac{1}{(t+1)^4} \left.\frac{u^{-2}}{-2}\right|_1^{x+1} dt$$

$$= -3\int_0^y \frac{1}{(t+1)^4}\left(\frac{1}{(x+1)^2} - \frac{1}{1^2}\right) dt$$

$$= 3\left(1 - \frac{1}{(x+1)^2}\right)\int_0^y \frac{1}{(t+1)^4}\, dt \qquad v = t+1 \quad dv = dt \qquad \begin{array}{c|c} t & v \\ \hline y & y+1 \\ 0 & 1 \end{array}$$

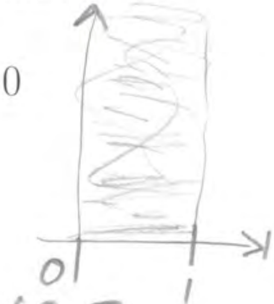
$$= 3\left(1 - \frac{1}{(x+1)^2}\right)\int_1^{y+1} v^{-4}\, dv$$

$$= \cancel{3}\left(1 - \frac{1}{(x+1)^2}\right)\left.\frac{v^{-3}}{-\cancel{3}}\right|_1^{y+1} = \left(1 - \frac{1}{(x+1)^2}\right)\left(1 - \frac{1}{(y+1)^3}\right)$$

So $$F_{XY}(x,y) = \begin{cases} \left(1 - \dfrac{1}{(x+1)^2}\right)\left(1 - \dfrac{1}{(y+1)^3}\right) & \text{for } x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

6. The continuous random variables X and Y have joint cumulative distribution function

$$F_{X,Y}(x,y) = \begin{cases} x^3 - x^3 e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 1 - e^{-y/4} & \text{for } x > 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



(a) What is $F_{X,Y}(\frac{1}{2}, 3)$? $= \frac{1}{8} - \frac{1}{8} e^{-3/4} = 0.0695$

(b) What is $F_{X,Y}(2, 3)$? $= 1 - e^{-3/4} = 0.5276$

(c) What is $F_{X,Y}(-1, 3)$? $= 0$

(d) What is $f_{X,Y}(x, y)$? For $0 \le x \le 1$ & $y \ge 0$ $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y} \frac{\partial}{\partial x} x^3 (1 - e^{-y/4})$

$= \frac{\partial}{\partial y} (1 - e^{-y/4}) 3x^2 = 3x^2 (-1) e^{-y/4} \cdot (-\frac{1}{4})$

$= 3x^2 \frac{1}{4} e^{-y/4}$

For $x > 1$ $y \ge 0$ $\frac{\partial}{\partial y} \frac{\partial}{\partial x} (1 - e^{-y/4}) = \frac{\partial}{\partial y} 0 = 0$

Otherwise, derivative of 0 is 0, so

$$f_{XY}(x,y) = \begin{cases} \frac{3}{4} x^2 e^{-y/4} & \text{for } 0 \le x \le 1 \text{ \& } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

7. Still for the joint distribution with

$$F_{X,Y}(x,y) = \begin{cases} x^3 - x^3 e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 1 - e^{-y/4} & \text{for } x > 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$f_{X,Y}(x,y) = \begin{cases} 3x^2 \underbrace{\tfrac{1}{4} e^{-y/4}} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Obtain $f_X(x)$ by integrating out y.

For $0 \le x \le 1$ $\quad f_X(x) = \int_0^\infty 3x^2 \frac{1}{4} e^{-y/4}\, dy$

$$= 3x^2 \underbrace{\int_0^\infty \frac{1}{4} e^{-y/4}\, dy}_{=1 \text{ Exponential}} = 3x^2, \text{ so}$$

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Calculate $F_X(x)$ by taking limits.

For $x < 0$, $\lim_{y \to \infty} F_{XY}(x,y) = \lim_{y\to\infty} 0 = 0$

For $x > 1$, $\quad '' \quad = \lim_{y \to \infty} (1 - e^{-y/4}) = 1 - \lim_{y \to \infty} \frac{1}{e^{y/4}} = 1$

For $0 \le x \le 1$, $\lim_{y \to \infty} F_{XY}(x,y) = \lim_{y \to \infty} x^3 (1 - e^{-y/4}) = x^3 \underbrace{\lim_{y \to \infty} (1 - e^{-y/4})}_{=1}$

$$\text{So } F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^3 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

(c) Obtain $f_X(x)$ from $F_X(x)$.

For $0 \le x \le 1$ $\quad f_X(x) = \frac{d}{dx} x^3 = 3x^2$, so

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

For $F_{X,Y}(x,y) = \begin{cases} x^3 - x^3 e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 1 - e^{-y/4} & \text{for } x > 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$ and

$f_{X,Y}(x,y) = \begin{cases} 3x^2 \frac{1}{4} e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$

(d) Obtain $f_Y(y)$ by integrating out x.

For $y \ge 0$ $f_Y(y) = \int_0^1 3x^2 \frac{1}{4} e^{-y/4}\, dx$

$= \frac{1}{4} e^{-y/4} \int_0^1 3x^2\, dx = \frac{1}{4} e^{-y/4} \cancel{3} \frac{x^3}{\cancel{3}} \Big|_0^1 = \frac{1}{4} e^{-y/4}$, so

$$f_Y(y) = \begin{cases} \frac{1}{4} e^{-y/4} & \text{for } y \ge 0 \\ 0 & \text{for } y < 0 \end{cases}$$

(e) Obtain $F_Y(y)$ by taking limits.

For $y \ge 0$ $\lim_{x \to \infty} F_{XY}(x,y) = \lim_{x \to \infty} \left(1 - e^{-y/4}\right) = \left(1 - e^{-y/4}\right)$, so

$$F_Y(y) = \begin{cases} 1 - e^{-y/4} & \text{for } y \ge 0 \\ 0 & \text{for } y < 0 \end{cases}$$

(f) Obtain $f_Y(y)$ from $F_Y(y)$.

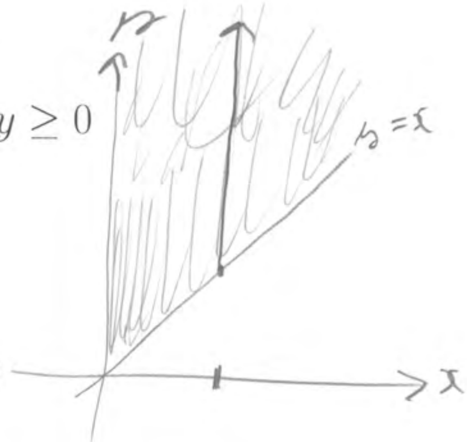
For $y \ge 0$, $F_Y'(y) = \frac{d}{dy}\left(1 - e^{-y/4}\right) = (-1) e^{-y/4} \left(-\frac{1}{4}\right)$

$= \frac{1}{4} e^{-y/4}$ so

$$f_Y(y) = \begin{cases} \frac{1}{4} e^{-y/4} & \text{for } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

8. Let $f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)} & \text{for } 0 \le x \le y \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$

Obtain $f_X(x)$.



For $x \ge 0$ $\quad f_X(x) = \int_x^\infty 2e^{-x}e^{-y}\,dy$

$$= 2e^{-x} \underbrace{\int_x^\infty e^{-y}\,dy}_{= e^{-x} \text{ cdf of Exponential}}$$

$u = -y \quad du = -dy$

y	u
∞	$-\infty$
x	$-x$

$$= 2e^{-x}(-1)\int_{-x}^{-\infty} e^u\,du$$

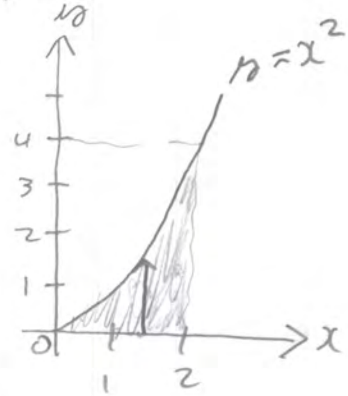
$$= 2e^{-x}\int_{-\infty}^{-x} e^u\,du = 2e^{-x}e^u\Big|_{-\infty}^{-x}$$

$$= 2e^{-x}\Big(e^{-x} - \underbrace{\lim_{u\to-\infty} e^u}_{=0}\Big) = 2e^{-2x}, \text{ so}$$

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

9. Let $f_{X,Y}(x,y) = \begin{cases} \frac{xy}{16} & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$

Find $P(Y < X^2)$. The answer is a number.



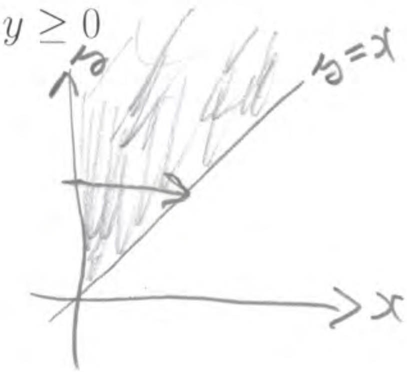
$$= \int_0^2 \int_0^{x^2} \frac{xy}{16}\, dy\, dx$$

$$= \frac{1}{16} \int_0^2 x \int_0^{x^2} y\, dy\, dx = \frac{1}{16} \int_0^2 x \left. \frac{y^2}{2} \right|_0^{x^2} dx$$

$$= \frac{1}{32} \int_0^2 x^5\, dx = \frac{1}{32} \left. \frac{x^6}{6} \right|_0^2 = \frac{1}{\cancel{32}} \frac{\cancel{32} \cdot 2}{6} = \frac{1}{3}$$

10. Let $f_{X,Y}(x,y) = \begin{cases} 4xy\, e^{-(x^2+y^2)} & \text{for } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Find $P(Y > X)$. The answer is a number.



$$\int_0^\infty \int_0^y 4xy\, e^{-x^2} e^{-y^2}\, dx\, dy$$

$$= 4\int_0^\infty y e^{-y^2} \int_0^y x e^{-x^2}\, dx\, dy$$

$u = x^2 \quad du = 2x\,dx$

x	u
y	y^2
0	0

$$= 2\int_0^\infty y\, e^{-y^2} \left(\int_0^{y^2} e^{-u}\, du \right) dy$$

$v = 1 - e^{-y^2}$

$dv = (-1)\, e^{-y^2} (-2y)$

$$= 2\int_0^\infty y\, e^{-y^2} \left(1 - e^{-y^2} \right) dy$$

y	
∞	1
0	0

$$= \int_0^1 v\, dv = \frac{v^2}{2}\Big|_0^1 = \frac{1}{2}$$