Sample Questions: Independence

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1. A jar contains 5 red balls and 15 black balls. Draw 2 balls randomly with replacement.

   (a) What is the probability that the first ball is red and the second is black? The answer is a number.

   $$\frac{5}{20} \cdot \frac{15}{20} = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$$

   (b) What is the probability of one red and one black in any order? The answer is a number.

   $$P(R_1 \cap B_2) + P(B_1 \cap R_2)$$
   $$= \frac{3}{16} + \frac{3}{16} = \frac{3}{8}$$

2. Let A and B be events of positive probability. Is it possible for A and B to be both disjoint and independent? Answer Yes or No and prove your answer.

No. A & B disjoint means $A \cap B = \emptyset$

So $P(A \cap B) = P(\emptyset) = 0 \neq P(A)P(B)$

Because $P(A) > 0$ & $P(B) > 0$ ■

3. Roll a fair die n times.

(a) What is the probability of observing at least one 4? $1 - P(\text{No } 4\text{s})$

$$= 1 - \left(\frac{5}{6}\right)^n$$

(b) How many times must you roll the die for the probability of at least one 4 to be 0.90 or more? The answer is a number.

$$1 - \left(\frac{5}{6}\right)^n \geq 0.9 \iff \frac{1}{10} \geq \left(\frac{5}{6}\right)^n$$

$$\iff \ln\left(\frac{1}{10}\right) \geq n \ln\left(\frac{5}{6}\right) \iff \frac{\ln(1/10)}{\ln(5/6)} \leq n$$

$$n \geq \frac{\ln\left(\frac{1}{10}\right)}{\ln(5/6)} = 12.63$$

So $n \geq 13$

4. A biased coin has $P(\text{Head}) = p$. Toss it three times.

   (a) List the elements of the sample space, along with their probabilities.

HHH	$p^3 (1-p)^0$
→ HHT	$p^2 (1-p)$
→ HTH	$p^2 (1-p)$
HTT	$p (1-p)^2$
→ THH	$p^2 (1-p)$
THT	$p (1-p)^2$
TTH	$p (1-p)^2$
TTT	$p^0 (1-p)^3$

   (b) What is $P(\text{Two Heads})$?

$$3 p^2 (1-p)$$

5. It is clear from the last problem that the probability of a string with k heads is the same, regardless of their placement. Suppose we toss the biased coin n times. What is the probability of k heads (for $k = 0, \ldots, n$)?

$$\binom{n}{k} p^k (1-p)^{n-k}$$

6. Again, a biased coin has $P(\text{Head}) = q$. Toss it until the first head occurs, and then stop.

(a) What is the probability that the first head appears on the fifth toss?

$$(1-q)^4 q$$

(b) What is the probability that a head eventually occurs (on toss 1 or 2 or ...)?

$$\sum_{k=1}^{\infty}(1-q)^{k-1} q$$

"Recall" $\displaystyle\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a}$

$$= \frac{q}{1-q} \sum_{\substack{k=1 \\ \uparrow \\ j}}^{\infty} (\underbrace{1-q}_{a})^k = \frac{q}{\cancel{1-q}} \cdot \frac{\cancel{1-q}}{(1-(1-q))} = \frac{q}{q} = 1$$

(c) What is the probability that the first head occurs on an even numbered toss (toss 2 or 4 or ...)?

$$\frac{a^j}{1-a} = \sum_{k=j}^{\infty} a^k$$

$$(1-q)q + (1-q)^3 q + (1-q)^5 q + \cdots$$

$$= \sum_{k=1}^{\infty} (1-q)^{2k-1} q = \sum_{k=1}^{\infty} \frac{(1-q)^{2k}}{(1-q)} q$$

$$= \frac{q}{1-q} \sum_{k=1}^{\infty} \left((1-q)^2\right)^k$$

$$a = (1-q)^2$$

$$= \frac{q}{\cancel{1-q}} \frac{(1-q)^{\cancel{2}}}{1-(1-q)^2} = \frac{q(1-q)}{1-(q^2-2q+1)}$$

$$= \frac{q(1-q)}{\cancel{1} - q^2 + 2q \cancel{-1}} = \frac{\cancel{q}(1-q)}{\cancel{q}(2-q)}$$

$$= \frac{1-q}{2-q}$$

7. In repeated tosses of a coin with $P(\text{Head}) = \theta$, what is the probability that the third head comes on the seventh toss?

theta

First 2H & 4T, no particular order
then H

By Prob. 5,

$$\underbrace{\binom{6}{2}\theta^2(1-\theta)^4}\cdot\theta$$

$$= \binom{6}{2}\theta^3(1-\theta)^4$$
