Foundations of Probability[1]
(Sections 1.2 and 1.3 in the text)

STA 256: Fall 2019

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The idea

Informally, a probability is a number between zero and one indicating how likely an event is to occur.

- Sample space S is the set of all things that can happen.
- Elements $s \in S$ are called *outcomes*.
- Subsets $A \subseteq S$ are called *events*.

Basic Properties of Probability

p. 5 in text

A probability measure is a function P from subsets of S to the real numbers, satisfying

1. $0 \leq P(A) \leq 1$
2. $P(\emptyset) = 0$
3. $P(S) = 1$
4. If $A_1, A_2 \ldots$ are disjoint subsets of S, $P\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$.

A Boring Example

Sell 500 lottery tickets, pick the winning number.

- $S = \{1, 2, \dots, 500\}$
- $P(\{2\}) = 1/500$
- $P(\text{Even Number}) = 1/2$

“Basic Properties” are really axioms (Kolmogorov, 1933)

The properties are a little redundant

Basic Properties

1. $0 \leq P(A) \leq 1$
2. $P(\emptyset) = 0$
3. $P(S) = 1$
4. If $A_1, A_2 \ldots$ are disjoint subsets of S, $P\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$.

Axioms

1. $P(A) \geq 0$ for any $A \subseteq S$
2. $P(S) = 1$
3. If $A_1, A_2 \ldots$ are disjoint subsets of S, $P\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$

Starting from the axioms, one can show

- $P(\emptyset) = 0$.
- If $A_1, \ldots, A_n$ are disjoint, $P\left(\cup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$ (finite additivity).
- Then it's smooth sailing.
- In this course, we will start with the 4 properties, and assume that Property 4 (additivity) applies to either finite or infinite collections of sets.

Some Elementary Theorems

- $P(A^c) = 1 - P(A)$
- If $A \subseteq B$ then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Inclusion-exclusion principle)

Some Not-So-Elementary Theorems

- Law of total Probability: Let $A_1, A_2, \ldots$ form a partition of the sample space S, and let B be any event. Then $P(B) = \sum_{k=1}^{\infty}(A_k \cap B)$.
- Sub-additivity: Let $A_1, A_2, \ldots$ be a collection of events, not necessarily disjoint. Then $P\left(\cup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} P(A_k)$
- Continuity 1: Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and let $A = \cup_{k=1}^{\infty} A_k$. Then $\lim_{k\to\infty} P(A_k) = P(A)$.
- Continuity 2: Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and let $A = \cap_{k=1}^{\infty} A_k$. Then $\lim_{k\to\infty} P(A_k) = P(A)$.

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