Sample Questions: Expected Value, Variance and Covariance

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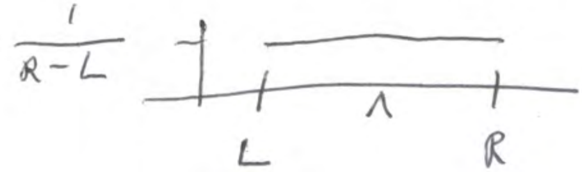
1. Let X have a continuous uniform distribution on (L, R). Calculate $E(X)$.



$$E(X) = \int_L^R x \, \frac{1}{R-L} \, dx = \frac{1}{R-L} \int_L^R x \, dx$$

$$= \frac{1}{R-L} \, \frac{x^2}{2} \Big|_L^R = \frac{1}{2(R-L)} (R^2 - L^2)$$

$$= \frac{1}{2\cancel{(R-L)}} (R+L)\cancel{(R-L)} = \frac{R+L}{2}$$

2. Recall that a fair game is one with expected value zero. You wager one dollar, and toss a coin with $P(\text{Head}) = \theta$. If it's heads, you win. In dollars, what should the payoff be so that the game is fair?

$$E(X) = \sum_x x\, P_X(x) \overset{\text{Fair}}{=} 0 = (-1)(1-\theta) + p\,\theta$$

$$\Rightarrow 1 - \theta = p\,\theta \Rightarrow p = \frac{1-\theta}{\theta}$$

3. Let $X \sim \text{Poisson}(\lambda)$. Calculate $E(X)$.

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \cancel{x} \frac{e^{-\lambda}\lambda^x}{\cancel{x}(x-1)!}$$

Let $k = x-1$
$x = k+1$

$$\begin{array}{c|c} x & k = x-1 \\ \hline \infty & \infty \\ 1 & 0 \end{array}$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k+1}}{k!} = \lambda \underbrace{\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}}_{=1}$$

$$= \lambda$$

4. Let the continuous random variable X have density $f_X(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$
where $\alpha > 0$

(a) Verify that $f_X(x)$ integrates to one.

$$\int_1^\infty \alpha x^{-\alpha-1}\,dx = \alpha \frac{x^{-\alpha}}{-\alpha}\Big|_1^\infty$$

$$= (-1)\frac{1}{x^\alpha}\Big|_1^\infty = (-1)\Big(\underbrace{\lim_{x\to\infty}\frac{1}{x^\alpha}}_{=0} - \frac{1}{1^\alpha}\Big)$$

$$= (-1)(0-1) = 1$$

(b) Calculate $E(X)$. For what values of α does $E(X)$ exist?

$$E(X) = \int_1^\infty x\frac{\alpha}{x^{\alpha+1}}\,dx = \alpha\int_1^\infty \frac{1}{x^\alpha}\,dx$$

If $\alpha = 1$ $\quad \alpha\int_1^\infty \frac{1}{x}\,dx = \alpha \ln(x)\Big|_1^\infty$

$$= \alpha\Big(\underbrace{\lim_{x\to\infty}\ln x}_{\text{"}\infty\text{"}} - \ln(1)\Big)$$ $E(X)$ does not exist

$$E(X) = \alpha \int_1^\infty x^{-\alpha}\, dx = \alpha \left. \frac{x^{-\alpha+1}}{-\alpha+1} \right|_1^\infty$$

$$= \frac{\alpha}{1-\alpha} \left. \frac{1}{x^{\alpha-1}} \right|_1^\infty$$

$$= \frac{\alpha}{1-\alpha} \Bigg(\underbrace{\lim_{x\to\infty} \frac{1}{x^{\alpha-1}}}_{\substack{= 0 \text{ if } \alpha > 1 \\ = \text{"}\infty\text{"} \ \alpha < 1}} - \frac{1}{1^{\alpha-1}} \Bigg)$$

If $\alpha > 1$, $E(X) = \frac{\alpha}{1-\alpha}(0-1)$

$$= \frac{\alpha}{\alpha - 1}$$

If $\alpha \le 1$ $E(X)$ does not exist

$(=\infty)$

5. Let $X \sim N(\mu, \sigma)$. Calculate $E(X)$. $= \mu$

$$E(x) \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$z = \frac{x-\mu}{\sigma}$$
$$x = \sigma z + \mu$$
$$dx = \sigma dz$$

$$= \int_{-\infty}^{\infty} (\sigma z + \mu) \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cancel{\sigma} dz$$

x	z
∞	∞
$-\infty$	$-\infty$

$$= \sigma \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \underbrace{\int_{-\infty}^{\infty} \mu f_z(z)\, dz}_{\mu}$$

$$= \mu + \sigma \underbrace{\int_{-\infty}^{0} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz}_{} + \sigma \int_{0}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$y = -z$$
$$dy = -dz$$

z	y
0	0
$-\infty$	∞

$$= \mu + \sigma \int_{\infty}^{0} (+y) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} (+1) dy + \sigma \int_{0}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \mu - \sigma \underbrace{\int_{0}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy + \sigma \int_{0}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz}_{0}$$

$= \mu$

6. Let X have a Gamma distribution with parameters α and λ. Calculate $E(X^k)$.

$$E(X^k) = \int_0^\infty x^k \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\lambda^{\alpha+k}} \underbrace{\int_0^\infty \frac{\lambda^{\alpha+k}}{\Gamma(\alpha+k)} e^{-\lambda x} x^{\alpha+k-1}\, dx}_{=1}$$

$$= \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)}$$

$$Y = bX$$

7. Prove $Var(bX) = b^2 Var(X)$. $Var(Y) = E\left[(Y - E(Y))^2\right]$

$$E(Y) = E(bX)$$
$$= \cancel{b}\, b\,E(X)$$

$$Var(bX) = E\left[(bX - b\mu_x)^2\right]$$
$$= E\left[(b(X-\mu_x))^2\right]$$
$$= E\left[b^2(X-\mu_x)^2\right] = b^2 E\left[(X-\mu_x)^2\right]$$
$$= b^2 Var(X)$$

8. Show $Var(X) = E(X^2) - [E(X)]^2$.

$$Var(X) = E\left[(X-\mu_X)^2\right]$$

$$= E(X^2 - 2X\mu_X + \mu_X^2)$$

$$= E(X^2) - 2\mu_X E(X) + E(\mu_X^2)$$

$$= E(X^2) - 2\mu_X^2 + \mu_X^2$$

$$= E(X^2) - \mu_X^2$$

$$= E(X)^2 - \left[E(X)\right]^2$$

9. Let X have density e^{-x} for $x \geq 0$ and zero otherwise. Calculate $Var(X)$.

This is a special case of Gamma (α, λ) with $\alpha = \lambda = 1$. From Problem 6, have

$$E(X^k) = \frac{\Gamma(\alpha + k)}{\lambda^k \Gamma(\alpha)}.$$

So for ~~[illegible]~~ ~~[illegible]~~ $\alpha = \lambda = 1$, $E(X^k) = \frac{\Gamma(k+1)}{1^k \Gamma(1)} = k!$

$E(X) = 1! = 1$

$E(X^2) = 2! = 2$, so

$$Var(X) = E(X^2) - E(X)^2$$

$$= 2 - 1 = 1$$

10 Let $X \sim N(\mu, \sigma^2)$. Calculate $\mathrm{Var}(X)$. Have $E(X) = \mu$ from Q5.

$$\mathrm{Var}(X) = E\left((X-\mu)^2\right) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \sigma^2 \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \left(\frac{x-\mu}{\sigma}\right)$ $\quad dz = \frac{1}{\sigma} dx$

x	z
∞	∞
$-\infty$	$-\infty$

$$= \sigma^2 \int_{-\infty}^{\infty} z^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$\overset{\text{symmetry}}{=} 2\sigma^2 \int_0^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$u = \frac{1}{2}z^2 \iff$

$z^2 = 2u \iff$

$z = \sqrt{2}\, u^{1/2}$

$dz = \sqrt{2} \cdot \frac{1}{2} u^{-\frac{1}{2}} du$

$$= 2\sigma^2 \int_0^{\infty} \cancel{2}u \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} e^{-u} \cancel{\sqrt{2}}\, \cancel{\tfrac{1}{2}} u^{-\frac{1}{2}} du$$

z	u
∞	∞
0	0

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{\frac{1}{2}} du$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{3/2 - 1} du = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \cdot \cancel{\tfrac{1}{2}} \cancel{\Gamma\left(\tfrac{1}{2}\right)} = \sigma^2$$

5/10

11. Show $Cov(X, Y) = E(XY) - E(X)E(Y)$.

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= E(XY - X\mu_Y - \mu_X Y + \mu_X \mu_Y)$$

$$= E(XY) - \mu_Y E(X) - \mu_X E(Y) + E(\mu_X \mu_Y)$$

$$= E(XY) - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y$$

$$= E(XY) - \mu_X \mu_Y$$

$$= E(XY) - E(X)E(Y)$$

12. Let X and Y be independent (continuous) random variables. Show $E(XY) = E(X)E(Y)$.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy$$

$$\overset{ind}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y) \int_{-\infty}^{\infty} x\, f_X(x)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y)\, E(X)\,dy$$

A constant

$$= E(X)\int_{-\infty}^{\infty} y\, f_Y(y)\,dy = E(X)E(Y)$$

13. If X and Y are independent, $Cov(X, Y) =$ 0

Because $cov(X,Y) = E(XY) - E(X)E(Y) = 0$