Sample Questions: Discrete Random Variables

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1. Roll two fair dice. Let X denote the sum of the two numbers.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(a) What is $p_X(12)$? The answer is a number. $P(X=12) = \frac{1}{36}$

(b) What is $F_X(12)$? The answer is a number. $P(X \leq 12) = 1$

(c) What is $p_X(27)$? The answer is a number. $P(27) = 0$

(d) What is $F_X(27)$? The answer is a number. $P(X \leq 27) = 1$

(e) What is $p_X(4)$? The answer is a number. $\frac{3}{36} = \frac{1}{12}$

(f) What is $F_X(4)$? The answer is a number. $P(X \leq 4) = P(X=2) + P(X=3) + P(X=4)$

$6/36 = 1/6$

(g) What is $F_X(4.5)$? The answer is a number.

$P(X \leq 4.5) = 1/6$

(h) What is $p_X(4.5)$? The answer is a number. $P(X = 4.5) = 6$

2. A biased coin has $P(\text{Head}) = \frac{1}{3}$. Toss it twice.

(a) List the elements of the sample space S, together with their probabilities.

ω	HH	HT	TH	TT
$P\{\omega\}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{3}\cdot\frac{2}{4}=\frac{4}{9}$

(b) Let X equal the number of heads. For what values of x is $P(X = x) > 0$?

$0,\ 1,\ 2$

(c) Give $p_X(x)$ and $F_X(x)$ just for $x = 0, 1, 2$.

x	$p(x)$	$F(x) = P(X \le x)$
0	4/9	4/9
1	4/9	8/9
2	1/9	9/9 = 1

(d) What is $p_X(1.5)$? $= 0$

(e) What is $F_X(1.5)$? $= F(1) = 8/9$

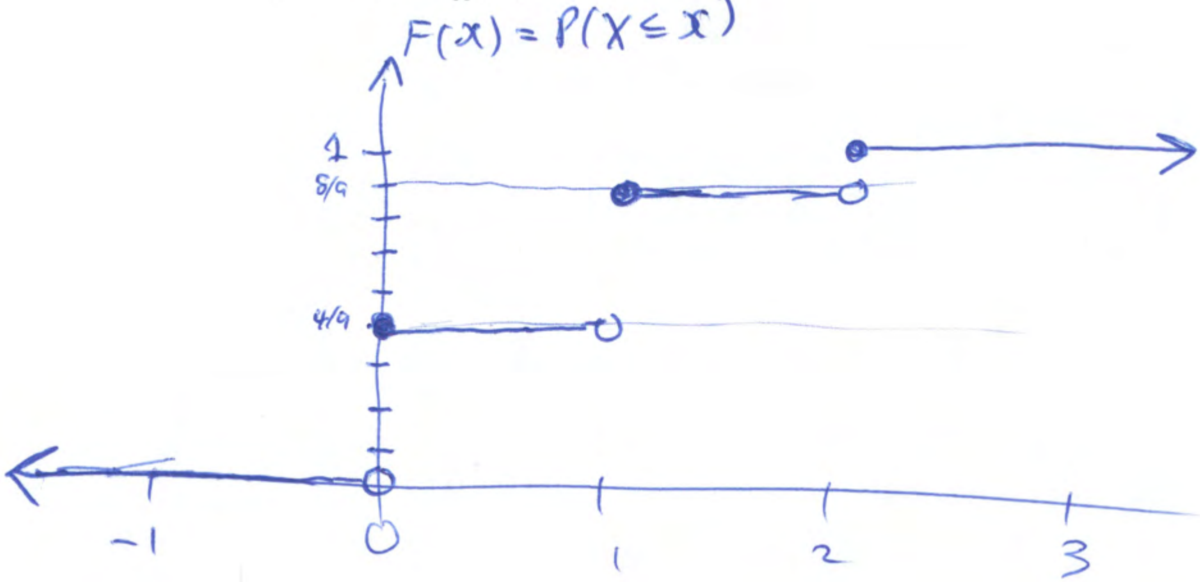
(f) What is $p_X(-9)$? $= 0$

(g) What is $F_X(-9)$? $= 0$

(h) What is $p_X(114)$? $= 0$

(i) What is $F_X(114)$? $= P(X=0)+P(X=1)+P(X=2) = 1$

(j) Graph $F_X(x)$.



$X(s)$

3. For a general random variable X, prove $\lim_{x\to\infty} F_X(x) = 1$.

Let $A_1 = \{s \in S : X(s) \le 1\}$ $\quad S = \bigcup_{k=1}^{\infty} A_k$, disjoint

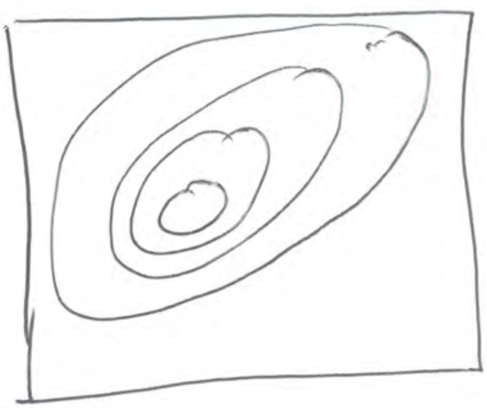
$A_2 = \{s \in S : 1 < X \le 2\}$

$A_3 = \{s \in S ; 2 < X \le 3\}$ $\quad$ so $P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k) = 1 = P(S)$

$\vdots$

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} F(n) = \lim_{n\to\infty} P(X \le n) = \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k)$$

$$= \sum_{k=1}^{\infty} P(A_k) = P(S) = 1$$



Let $A_n = \{s \in S : X(s) \le n\}$

$A_1 \subseteq A_2 \subseteq A_3 \cdots$

$\bigcup_{k=1}^{\infty} A_k = S$

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} F(n) = \lim_{n\to\infty} P(X \le n) = \lim_{n\to\infty} P(A_n)$$

$$= P(S) = 1$$

4. For a general random variable X, prove $\lim_{x \to -\infty} F_X(x) = 0$.

Let $A_n = \{ s \in S : X(s) \leq -n \}$

$A_1 \supseteq A_2 \supseteq A_3 \cdots \quad \bigcap_{k=1}^{\infty} A_k = \emptyset$

$\lim_{x \to -\infty} F(x) = \lim_{n \to \infty} F(-n) = \lim_{n \to \infty} P(A_n) = P(\emptyset) = 0$

By ~~that~~ a nested set HW problems

5. Let the discrete random variable X have probability mass function $p_X(x) = cx$ for $x = 1, 2, 3, 4$ and zero otherwise. What is the constant c?

$$1 = \sum_{x=1}^{4} P_X(x) = \sum_{x=1}^{4} cx = c\sum_{x=1}^{4} x \iff c = \frac{1}{\sum_{x=1}^{4} x}$$

$$= \frac{1}{1+2+3+4} = \frac{1}{10}$$

6. Prove that the binomial probabilities sum to one. The formula sheet for Test 2 will have the Binomial Theorem: $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$

$$P_X(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

$$a = \theta,\ b = 1-\theta$$

$$\sum_{x=0}^{n}\binom{n}{x}\theta^x(1-\theta)^{n-x} = (\theta + 1 - \theta)^n$$

$$= 1^n = 1$$

7. Let X have a binomial distribution with $n = 5$ and $\theta = \frac{1}{4}$. $p(x) = \binom{5}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{n-x}$

(a) What is $p_X(0)$? The answer is a number.

$$= \binom{5}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{5-0} = \left(3/4\right)^5 = 0.2373$$

(b) What is $F_X(0)$? The answer is a number.

$$F(0) = p(0) = 0.2373$$

(c) What is $F_X(5)$? The answer is a number.

$$P(X \leq 5) = 1$$

(d) What is $p_X(2)$? The answer is a number.

$$\binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 = \frac{5!}{2!3!} \quad \frac{27}{1024} \simeq 0.2367$$

(e) What is $F_X(1)$? The answer is a number.

$$F_X(1) = p(0) + p(1),\quad p(1) = \binom{5}{1}\frac{1}{4}\left(\frac{3}{4}\right)^4 \doteq 0.3955$$

$$F(1) = p(0) + p(1) = 0.2373 + 0.3955$$

$$\simeq 0.6328$$

8. Cheap umbrellas are shipped to the dollar store in boxes of 20. The probability that the umbrella is defective (you can't even use it once) is 0.10. We will assume that being defective or not for the 20 umbrellas in a box are independent events, though this assumption may not be safe in practice, depending on the manufacturing and shipping process.

   (a) What is the probability that all 20 umbrellas are okay? The answer is a number.

   $$.9^{20} = 0.1216$$

   (b) Obtain that last number as $p_X(0)$ for one of the standard probability distributions.

   $$X \sim \text{Binomial}(20, \tfrac{1}{10}) \qquad p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

   $$P_X(0) = \binom{20}{0}\left(\frac{1}{10}\right)^0\left(1-\frac{1}{10}\right)^{20-0}$$

   (c) That is the probability that exactly two umbrellas are defective? The answer is a number.

   $$P_X(2) = \binom{20}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18} = 0.2852$$

   (d) What is the probability that two or fewer umbrellas are defective? The answer is a number.

   $$P(X \leq 2) = P(0) + P(1) + P(2)$$

   $$P(1) = \binom{20}{1}\frac{1}{10}\left(\frac{9}{10}\right)^{19} = 0.2702$$

   $$P(X \leq 2) = 0.1216 + 0.2702 + 0.2852$$

   $$= 0.677$$

9. In a box of 20 umbrellas, 2 are defective. If you sample 5 umbrellas randomly without replacement, what is the probability of at least one defective? The answer is a number.

$$1 - P(\text{All okay})$$
$$= 1 - \frac{\binom{18}{5}}{\binom{20}{5}}$$

10. Going back to the assumptions of Question 8, the probability of a defective umbrella is 0.10, they are independent, and they are shipped in boxes of 20. You choose a box at random, and then sample 5 umbrellas randomly without replacement, what is the probability of at least one defective? The answer is a number.

$$1 - P(\text{All okay})$$

$$P(\text{All okay}) = \sum_{k=0}^{20} P(\text{All ok} \mid k \text{ defect.})\, P(k \text{ def.})$$

$$= \sum_{k=0}^{15} \frac{\binom{20-k}{5}}{\binom{20}{5}} \binom{20}{k} \theta^k (1-\theta)^{20-k}$$

$$= \sum_{k=0}^{15} \frac{\frac{\cancel{(20-k)!}}{\cancel{5!}(20-k-5)!}}{\frac{\cancel{20!}}{\cancel{5!}(20-5)!}} \frac{\cancel{20!}}{k!\,\cancel{(20-k)!}} \theta^k (1-\theta)^{20-k}$$

$$= \sum_{k=0}^{15} \frac{15!}{k!\,(15-k)!} \theta^k (1-\theta)^{20-k}$$

$$= \underbrace{\sum_{k=0}^{15} \binom{15}{k} \theta^k (1-\theta)^{15-k}}_{=1} \cdot (1-\theta)^5$$

$$= 0.9^5$$

At least one bad

$$P(\text{At least one bad}) = 1 - .9^5 = 0.4095$$

11. It is true love, but still the chances your significant other will break up with you on any given day is a tenth of one percent. Assuming 365 days in a year and independence, what is the probability that your relationship will last at least one year? The answer is a number.

$X = \#$ of happy days before breakup

$X \sim G(\theta = 0.001)$ $p(x) = (1-\theta)^x \theta$

And $\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a}$

$$P(X \geq 365) = \sum_{x=365}^{\infty} (1-\theta)^x \theta$$

$$= \theta \sum_{x=365}^{\infty} (1-\theta)^x = \cancel{\theta} \frac{(1-\theta)^{365}}{1-\cancel{(1-\theta)}}$$

$$= 0.999^{365} = 0.694$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

12. Let X_n have a binomial distribution with parameters n and θ_n. That is, $p_{X_n}(x) = \binom{n}{x}\theta_n^x(1-\theta_n)^{n-x}$ for $x = 0, \ldots, n$. The probability θ_n depends on n. As $n \to \infty$, $\theta_n \to 0$ in such a way that $n\theta_n = \lambda$ remains constant. Find $\lim_{n\to\infty} p_{X_n}(x)$, and identify the resultng distribution by name.

$$n\theta_n = \lambda \Leftrightarrow \theta_n = \frac{\lambda}{n}$$

$$\lim_{n\to\infty}\binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty}\binom{n}{x}\left(\frac{\lambda/n}{1-\lambda/n}\right)^x\left(1-\frac{\lambda}{n}\right)^n$$

$$= \lim_{n\to\infty}\frac{n!}{x!(n-x)!}\,\frac{\lambda^x}{(n-\lambda)^x}\left(1-\frac{\lambda}{n}\right)^n$$

$$\lim_{n\to\infty}\frac{\lambda^x}{x!}\,\underbrace{n\,(n-1)\cdots(n-x+1)}_{x\text{ terms}}$$

$$\frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n(n-1)\cdots(n-x+1)}{(n-\lambda)^x}\left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n}{n-\lambda}\lim_{n\to\infty}\frac{n-1}{n-\lambda}\cdots\lim_{n\to\infty}\frac{n-x+1}{n-\lambda}\cdot\underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}}$$

$$= \frac{e^{-\lambda}\lambda^x}{x!}$$ Poisson