Sample Questions: Counting Methods for Computing Probabilities

STA256 Fall 2019. Copyright information is at the end of the last page.

1. Ten students are standing in line. If they lined up completely at random, what is the probability that Romeo is standing next to Juliet? This question was on the 2018 final exam.

$$\frac{9!\,2}{10!} = \frac{2}{10} = \frac{1}{5}$$

2. A deck of 52 cards is shuffled thoroughly. What is the probability that the four aces are all next to each other?

$$\frac{49!\,4!}{52!}$$

3. A standard deck of 52 cards has four "suits:" spades, diamonds, hearts and clubs. Within each suit, the face values of the 13 cards are 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace. A "hand" of poker is 5 cards, selected randomly without replacement.

   (a) A "flush" is a hand with 5 cards all of the same suit. What is the probability of a flush?

$$\frac{\binom{13}{5} \cdot 4}{\binom{52}{5}}$$

   (b) A "straight" is a hand in which the 5 cards are in sequence. Suit is ignored. An Ace can be either high or low. What is the probability of a straight?

A 2 3 4 5 6 7 8 9 10 J Q K A

$$\frac{10 \cdot 4^5}{\binom{52}{5}}$$

$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

4. Using the formula for $\binom{n}{k}$ from the formula sheet, and the Multiplication Principle, prove that the number of ways that n objects can be divided into k subsets with n_i objects in set i is $\binom{n}{n_1 \cdots n_k} = \frac{n!}{n_1! \cdots n_k!}$.

There are $\binom{n}{n_1}$ ways to choose objects in first subset

" " $\binom{n-n_1}{n_2}$ " " " " " 2nd "

" " $\binom{n-n_1-n_2}{n_3}$ " " " " " 3d "

⋮

" " $\binom{n-n_1-\cdots-n_{k-1}}{n_k}$ " " " " " kth "

By Multiplication principle, the total # of ways

$$\frac{n!}{n_1!\cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)!}}{n_2!\cancel{(n-n_1-n_2)!}} \cdot \frac{\cancel{(n-n_1-n_2)!}}{n_3!\cancel{(n-n_1-n_2-n_3)!}} \cdots \frac{\cancel{(n-n_1-\cdots-n_{k-1})!}}{n_k!\,0!}$$

$$= \frac{n!}{n_1! \cdots n_k!} = \binom{n}{n_1 \cdots n_k}$$

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

5. In how many ways can 20 basketball players be divided into 4 teams of 5?

$$\binom{20}{5\quad 5\quad 5\quad 5} = \frac{20!}{5!\,5!\,5!\,5!}$$

6. In how many ways can 6 red flags, 2 blue flags and 4 yellow flags be arranged? The flags are indistinguishable.

$$\binom{12}{6\quad 2\quad 4}$$

7. In a game of poker, four players are each dealt 5 cards from a 52-card deck. How many ways are there to deal the cards?

$$\cancel{\binom{52}{5}}\binom{52}{5\quad 5\quad 5\quad 5\quad 32}$$

8. Sample k balls from a jar containing n numbered balls. How many outcomes are there is the sampling is

(a) With replacement?

$$n^k$$

(b) Without replacement?

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

9. A jar contains 10 red balls and 20 blue balls. If 5 balls are randomly sampled without replacement, what is the probability of

(a) All blue?

$$\frac{\binom{20}{5}}{\binom{30}{5}} = \frac{2584}{23751} \approx 0.109$$

(b) Two red and three blue?

$$\frac{\binom{10}{2} \cdot \binom{20}{3}}{\binom{30}{5}}$$

(c) At least one red?

$$1 - P(\text{All Blue}) = 1 - 0.109$$

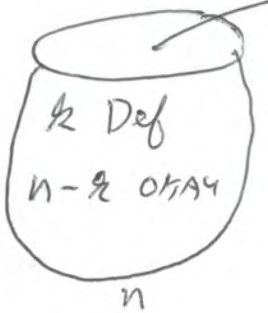
$$\approx 0.891$$

(d) A jar contains 10 red balls and 20 blue balls. If 5 balls are randomly sampled without replacement, what is the probability of obtaining k red balls, $k = 0, \ldots, 5$?

$$\frac{\binom{10}{k} \cdot \binom{20}{5-k}}{\binom{30}{5}}$$

If sample more than # ok, sure to get a defective

10. A shipment of n electronic components has k defectives. If we sample m components without replacement, what is the probability of observing at least one defective?



$m \leq n-k$

$P(\text{At least one Def})$

$1 - P(\text{All OKAY})$

$$\begin{cases} 1 - \dfrac{\binom{n-k}{m}}{\binom{n}{m}}, & m \leq n-k \\ 1, & m > n-k \end{cases}$$