Sample Questions: Continuous Random Variables

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1. The continuous random variable X has density $f(x) = \begin{cases} \frac{c}{x^{\alpha+1}} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}$ where $\alpha > 0$.

   (a) Find the constant c

$$1 = c\int_1^\infty \frac{1}{x^{\alpha+1}}\,dx = c\int_1^\infty x^{-\alpha-1}\,dx = c\left.\frac{x^{-\alpha}}{-\alpha}\right|_1^\infty$$

$$= \frac{-c}{\alpha}\left(\left.\frac{1}{x^\alpha}\right|_1^\infty\right) = \frac{-c}{\alpha}\left(\lim_{x\to\infty}\frac{1}{x^\alpha} - \frac{1}{1^\alpha}\right) = -\frac{c}{\alpha}(0-1)$$

$$= \frac{c}{\alpha} = 1 \text{ so } c = \alpha$$

   (b) Find the cumulative distribution function $F(x)$.

For $x \geq 1$, $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_1^x \frac{\alpha}{t^{\alpha+1}}\,dt$

$$= \alpha\int_1^x t^{-\alpha-1}\,dt = \alpha\left.\frac{t^{-\alpha}}{-\alpha}\right|_1^x = (-1)(x^{-\alpha} - 1^{-\alpha})$$

$$= 1 - \frac{1}{x^\alpha}, \text{ so}$$

$$F(x) = \begin{cases} 1 - \frac{1}{x^\alpha} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}$$

   (c) The median of this distribution is that point m for which $P(X \leq m) = \frac{1}{2}$. What is the median? The answer is a function of α.

$$\text{Set } F(m) = \frac{1}{2} = 1 - \frac{1}{m^\alpha} \iff \frac{1}{m^\alpha} = \frac{1}{2}$$

$$\iff m^\alpha = 2 \iff m = 2^{\frac{1}{\alpha}}$$

2. Let $F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^\theta & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$

(a) If $\theta = 3$, what is $P\left(\frac{1}{2} < X \le 4\right)$? The answer is a number.

$F(4) - F(\tfrac{1}{2}) = 1 - \left(\tfrac{1}{2}\right)^3 = 1 - \tfrac{1}{8} = \frac{7}{8}$

(b) Find $f(x)$. $= F'(x)$ for $0 \le x \le 1$

$F'(x) = \theta x^{\theta - 1}$, so

$$f(x) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

3. The Uniform(L, R) distribution has density $f(x) = \begin{cases} \frac{1}{R-L} & \text{for } L \le x \le R \\ 0 & \text{Otherwise} \end{cases}$

(a) Give the cumulative distribution function.

For $L \le x \le R$

$$F(x) = \int_L^x \frac{1}{R-L}\, dt$$

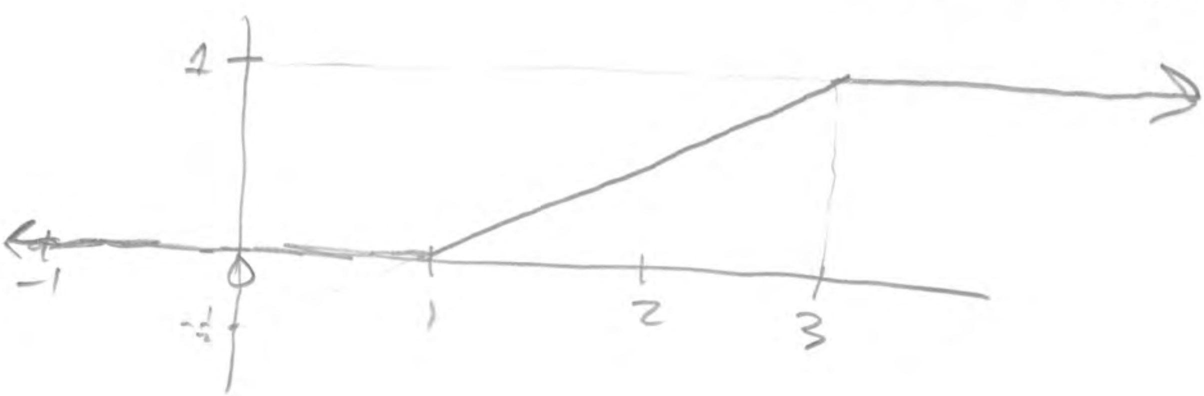
$$= \frac{1}{R-L}\, t \Big|_L^x = \frac{x-L}{R-L} \longrightarrow$$

AREA $= \frac{x-L}{R-L}$

$$F(x) = \begin{cases} 0 & \text{for } x < L \\ \frac{x-L}{R-L} & \text{for } L \le x \le R \\ 1 & \text{for } x > R \end{cases}$$

(b) Graph the cumulative distribution function. for $L = 1$, $R = 3$

for $1 \le x \le 3$ $\quad F(x) = \frac{x}{2} - \frac{1}{2}$



4. The Exponential(λ) distribution has density $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

(a) Show $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Let $u = -\lambda x \quad du = -\lambda\, dx$

$$-\int_0^\infty -\lambda e^{-\lambda x}\, dx$$

x	
∞	$-\infty$
0	0

$$= -\int_0^{-\infty} e^u\, du = \int_{-\infty}^0 e^u\, du = e^u \Big|_{-\infty}^0 = e^0 - \lim_{u \to -\infty} e^u$$

$$= 1 - 0 = 1$$

(b) Find $F(x)$

For $x \geq 0$, $F(x) = -\int_0^x -\lambda e^{-\lambda t}\, dt \quad u = -\lambda t \quad du = -\lambda\, dt$

t	u
x	$-\lambda x$
0	0

$$= -\int_0^{-\lambda x} e^u\, du = \int_{-\lambda x}^0 e^u\, du$$

$$= 1 - e^{-\lambda x}$$

, and

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt$$

5. The Gamma(α, λ) distribution has density $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x}\, x^{\alpha-1} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

Show $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

$$\int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}\, dx$$

$t = \lambda x \quad dt = \lambda dx$

$x = \frac{t}{\lambda}$

$dx = \frac{1}{\lambda} dt$

x	t
∞	∞
0	0

$$= \int_0^\infty \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} e^{-t} \left(\frac{t}{\cancel{\lambda}}\right)^{\alpha-1} \frac{1}{\cancel{\lambda}}\, dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1}\, dt = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

6. The Normal(μ, σ) distribution has density $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$

(a) Show that $f(x)$ is symmetric about μ, meaning $f(\mu + x) = f(\mu - x)$.

$$f(x+\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x+\cancel{\mu}-\cancel{\mu})^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2}$$

$$//$$

$$f(\mu-x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\cancel{\mu}-x\cancel{-\mu})^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2}$$

! $\longrightarrow$ (b) Let $X \sim \mathrm{N}(\mu, \sigma)$ and $Z = \frac{X-\mu}{\sigma}$. Find the density of Z.

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} P(Z \le z)$$

$$= \frac{d}{dz} P\left(\frac{X-\mu}{\sigma} \le z\right) \overset{\text{Isolate } X}{=} \frac{d}{dz} P(X-\mu \le \sigma z)$$

$$= \frac{d}{dz} P(X \le \sigma z + \mu) = \frac{d}{dz} F_X(\sigma z + \mu)$$

$$= f_X(\sigma z + \mu) \cdot \sigma$$

$$= \cancel{\sigma} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\sigma z + \cancel{\mu} - \cancel{\mu})^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\cancel{\sigma^2}}\cancel{\sigma^2} z^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$N(0, 1)$

$\mu = 0, \ \sigma^2 = 1$

7. Let $Z \sim N(0,1)$ (standard normal), so that $f_x(z) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. If $x > 0$, show $F_z(-x) = 1 - F_z(x)$.

$$F_z(-x) = -\int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} (-)dz$$

Let $t = -z$ $\quad dt = -dz$

z	t
$-x$	x
$-\infty$	∞

$$= -\int_{\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

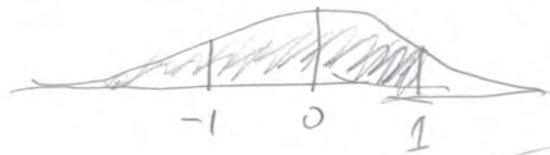
$$= 1 - F_z(x)$$ ✓

8. Let $X \sim N(\mu = 50, \sigma^2 = 100)$ $\quad Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

(a) Find $P(X < 60)$. The answer is a number.

$$P(X<60) = P(X-\mu < 60-50) = P\left(\frac{X-\mu}{\sigma} < \frac{10}{10}\right)$$

$$= P(Z<1)$$



$$1 - F_Z(-1) = 1 - .1587 = 0.8413$$

(b) Find $P(X > 30)$. The answer is a number.

$$P(X>30) = P\left(\frac{X-\mu}{\sigma} > \frac{30-50}{10}\right) = P(Z>-2)$$

$$= 1 - F_Z(-2)$$

$$= 1 - 0.0228 = 0.9772$$

(c) Find $P(30 < X < 55)$.

$$P\left(\frac{30-50}{10} < \frac{X-\mu}{\sigma} < \frac{55-50}{10}\right)$$

$$= P\left(-2 < Z < \frac{1}{2}\right)$$

$$= 1 - F_Z(-2) - F_Z\left(-\frac{1}{2}\right)$$

$$= 1 - 0.0228 - 0.3085$$

$$= 0.6687$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\,dt$$

9. Let $Z \sim N(0,1)$ (standard normal), so that $f_Z(z) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. Show $\int_{-\infty}^{\infty} f_Z(z) = 1$. Hint: Let $t = \frac{z^2}{2}$. You may use $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

// symmetry

$$2\int_0^\infty f_z(z)\,dz = \qquad t = \frac{z^2}{2} \iff z = \sqrt{2t} = \sqrt{2}\, t^{\frac{1}{2}}$$

$$dz = \sqrt{2}\,\frac{1}{2}\, t^{-\frac{1}{2}}\,dt$$

$$= \cancel{2}\int_0^\infty \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} e^{-t}\, \cancel{\sqrt{2}}\, \cancel{\tfrac{1}{2}}\, t^{-\frac{1}{2}}\,dt \qquad \begin{array}{c|c} z & t \\ \infty & \infty \\ 0 & 0 \end{array}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\,dt$$

$$= \frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt = \int_0^\infty e^{-t} dt = 1$$

10. The beta density with parameters α and β is $f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$

Let $X \sim \text{Beta}(\alpha, \beta)$ with $\beta = 1$.

(a) Write the density of X for $0 \le x \le 1$. Simplify. You will prove $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ in homework.

$$f_X(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1}(1-x)^{1-1} = \frac{\alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} x^{\alpha-1}$$
$$= \alpha x^{\alpha-1}$$

(b) Let $Y = 1/X$. The *support* of a distribution is the smallest (closed) set $A \subseteq \mathbb{R}$ with $P(X \in A) = 1$. What is the support of Y? Show your work.

$0 < X < 1 \iff \infty > \frac{1}{X} > 1$ that is

$1 < \frac{1}{X} < \infty$ or $1 \le Y < \infty$

(c) Derive $f_y(y)$. Don't forget to specify where the density is greater than zero.

"Distribution function technique"

For $y \ge 1$

$$f_Y(y) = \frac{d}{dy} F_X(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P\left(\frac{1}{X} \le y\right)$$
$$= \frac{d}{dy} P\left(X \ge \frac{1}{y}\right) = \frac{d}{dy}\left(1 - F_X(y^{-1})\right)$$
$$= -\frac{d}{dy} F_X(y^{-1}) = -f_X(y^{-1}) \cdot (-1) y^{-2}$$
$$= \alpha \left(\frac{1}{y}\right)^{\alpha-1} \cdot \frac{1}{y^2} = \frac{\alpha}{y^{\alpha+1}}, \text{ and}$$

$$f_Y(y) = \begin{cases} \frac{\alpha}{y^{\alpha+1}} & \text{for } y \ge 1 \\ 0 & \text{for } y < 1 \end{cases}$$

$$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

11. Let $Z \sim N(0,1)$ and $Y = Z^2$.

   (a) For what values of y is $f_y(y) > 0$?

   $-\infty < Z < \infty \quad Z^2 = Y \geq 0$

   (b) Show that Y has a gamma distribution and give the parameters. You may use the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, without proof.

For $y \geq 0$ $\quad f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(Z^2 \leq y)$

$$= \frac{d}{dy} P\left(|Z| \leq y^{\frac{1}{2}}\right)$$

$$= \frac{d}{dy} P\left(-y^{1/2} \leq Z \leq y^{1/2}\right) = \frac{d}{dy}\left(F_Z(y^{1/2}) - F_Z(-y^{\frac{1}{2}})\right)$$

$$= f_Z(y^{1/2}) \cdot \tfrac{1}{2} y^{-1/2} - f_Z(-y^{1/2}) \cdot (-1)\tfrac{1}{2} y^{-1/2}$$

$$= \frac{1}{2} y^{-\frac{1}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^{1/2})^2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-y^{1/2})^2} \right)$$

$$= \cancel{\tfrac{1}{2}} y^{-\frac{1}{2}} \cancel{2}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \right)$$

$$\left(\text{Looking for } \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda y} y^{\alpha - 1} \right)$$

$$= \frac{\left(\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}} e^{-\frac{1}{2}y} y^{1/2 - 1} = \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} e^{-\frac{1}{2}y} y^{\frac{1}{2}-1}$$

GAMMA$(\alpha = \frac{1}{2}, \lambda = \frac{1}{2})$ Chi-squared distribution with one df.

12. In this problem, the random variable X is transformed by its own distribution function. Let the continuous random vabriale X have distribution function $F_x(x)$, and let $Y = F_x(X)$. $F_x(x)$ strictly increasing on a single interval. $= P(X \leq x)$

(a) For what values of y is $f_y(y) > 0$? Hint: as x ranges from $-\infty$ to ∞, $F_x(x)$ ranges from 0 to 1.

(b) Find $f_y(y)$.

For $0 \leq y \leq 1$ $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y)$

$$= \frac{d}{dy} P(F_X(X) \leq y) = \frac{d}{dy} P\left(F_X^{-1}(F_X(X)) \leq F_X^{-1}(y)\right)$$

$$= \frac{d}{dy} P(X \leq F_X^{-1}(y)) = \frac{d}{dy} F_X\left(F_X^{-1}(y)\right) = \frac{d}{dy} y = 1$$

So

$$f_Y(y) = \begin{cases} 1 & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Uniform $(L = 0, R = 1)$
