Sample Questions: Conditional Distributions and Independent Random Variables

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1. Let X and Y be continuous random variables. Prove that X and Y are independent if and only if $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$.

Suppose X & Y are independent. Then $F_{XY}(x,y) = F_X(x)\,F_Y(y)$ and

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{XY}(x,y)$$

$$\overset{ind}{=} \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_X(x)\,F_Y(y) = \underbrace{\frac{\partial}{\partial x} F_X(x)}\;\underbrace{\frac{\partial}{\partial y} F_Y(y)}$$

$$= f_X(x)\, f_Y(y) \quad \blacksquare$$

Suppose $\underline{f_{XY}(x,y) = f_X(x)\, f_Y(y)}$. Then

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(s,t)\, dt\, ds$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(s)\, f_Y(t)\, dt\, ds$$

$$= \int_{-\infty}^{x} f_X(s) \underbrace{\int_{-\infty}^{y} f_Y(t)\, dt}\, ds = \int_{-\infty}^{x} f_X(s)\, F_Y(y)\, ds$$

$$= F_Y(y) \int_{-\infty}^{x} f_X(s)\, ds = F_X(x)\, F_Y(y) \quad \underline{\text{Independent}}.$$

2. Let X and Y be discrete random variables. Prove that if $p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$, then X and Y are independent.

$$F_{XY}(x,y) = \sum_{s \le x} \sum_{t \le y} P_{XY}(s,t)$$

$$= \sum_{s \le x} \sum_{t \le y} P_X(s) P_Y(t)$$

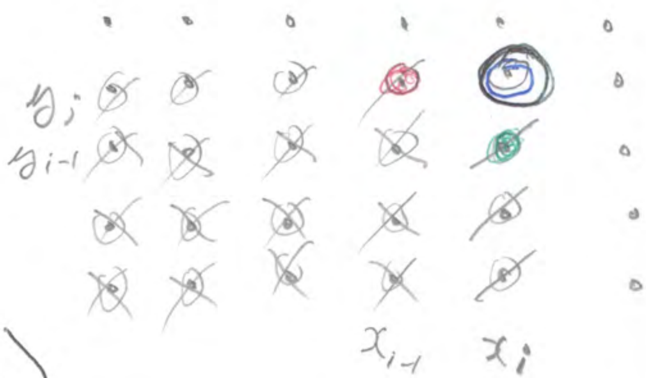
$$= \underbrace{\sum_{s \le x} P_X(s)}_{P(X \le x)} \underbrace{\sum_{t \le y} P_Y(t)}_{P(Y \le y)}$$

$$= F_X(x) F_Y(y)$$

Independent

3. Let X and Y be discrete random variables. Prove that if X and Y are independent, then $p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$.

Note $P_X(x_i) = F_X(x_i) - F_X(x_{i-1})$



$$P_{XY}(x_i, y_j)$$

$$= F_{XY}(x_i, y_j) - F_{XY}(x_{i-1}, y_j)$$

$$- F_{XY}(x_i, y_{j-1}) + F_{XY}(x_{i-1}, y_{j-1})$$

$$= F_X(x_i)\underbrace{F_Y(y_j)} - F_X(x_{i-1})\underbrace{F_Y(y_j)}$$

$$- F_X(x_i)\underbrace{F_Y(y_{j-1})} + F_X(x_{i-1})\underbrace{F_Y(y_{j-1})}$$

$$= F_Y(y_j)\left(F_X(x_i) - F_X(x_{i-1})\right)$$

$$- F_Y(y_{j-1})\left(F_X(x_i) - F_X(x_{i-1})\right)$$

$$= F_Y(y_j)\,P_X(x_i) - F_Y(y_{j-1})\,P_X(x_i)$$

$$= \underbrace{\left(F_Y(y_j) - F_Y(y_{j-1})\right)}_{P_Y(y_j)} P_X(x_i) = P_X(x_i)\,P_Y(y_j)$$

4. Let $p_{X,Y}(x,y) = \frac{|x-2y|}{\cancel{19}\ 20}$ for $x = 1, 2, 3$ and $y = 1, 2, 3$, and zero otherwise.

	$x=1$	$x=2$	$x=3$	
$y=3$	5/20	4/20	3/20	12/20
$y=2$	3/20	2/20	1/20	6/20
$y=1$	1/20	0/20	1/20	2/20
	9/20	6/20	5/20	1.00

(a) What is $p_{Y|X}(1|2)$?

$$= \frac{P(Y=1, X=2)}{P(X=2)} = \frac{0/20}{6/20} = 0$$

(b) What is $p_{X|Y}(1|2)$?

$$= \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{3/20}{6/20} = \frac{1}{2}$$

(c) Are X and Y independent? Answer Yes or No and prove your answer.

NO, Because $P_{X,Y}(2,1) = 0 \neq P_X(2)\, P_Y(1) = \frac{6}{20} \cdot \frac{2}{20}$

5. Let $f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)} & \text{for } 0 \le x \le y \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$

(a) Find $f_{X|Y}(x|y)$. $= \dfrac{f_{XY}(x,y)}{f_Y(y)}$

For $y \ge 0$

$$f_Y(y) = \int_0^y 2e^{-x}e^{-y}\,dx$$

$$= 2e^{-y}\int_0^y e^{-x}\,dx = 2e^{-y}(1-e^{-y}) \quad (\text{cdf of Exp}(1))$$

So for $0 \le x \le y$ $f_{X|Y}(x|y) = \dfrac{\cancel{2}e^{-x}\cancel{e^{-y}}}{\cancel{2}\cancel{e^{-y}}(1-e^{-y})}$, and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{e^{-x}}{1-e^{-y}} & \text{for } 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

(b) Are X and Y independent? Answer Yes or No and prove your answer.

(NO, Because $\underset{\overset{\|}{0}}{f_{XY}(3,1)} = f_X(3)\,f_Y(1)$) or

For $x \ge 0$

$$f_X(x) = \int_x^\infty 2e^{-x}e^{-y}\,dy = 2e^{-x}\int_x^\infty e^{-y}\,dy = 2e^{-x}e^{-x}$$

$= 2e^{-2x}$ Answer is no Bec

$$0 = f_{XY}(3,1) \ne f_X(3)\,f_Y(1) = \underbrace{2e^{-3}\cdot 2e^{-1}(1-e^{-1})}$$

6. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f(x)$ and cumulative distribution function $F(x)$. Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_Y(y)$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(\max_i X_i \le y)$$

$$= \frac{d}{dy} P(\text{All } X_i \le y) = \frac{d}{dy} P\left(\bigcap_{i=1}^{n} \{X_i \le y\}\right)$$

$$= \frac{d}{dy} \prod_{i=1}^{n} P(X_i \le y) = \frac{d}{dy} \prod_{i=1}^{n} F_X(y)$$

$$= \frac{d}{dy} \left(F_X(y)\right)^n = n\, F_X(y)^{n-1} f_X(y)$$

$$\int_{-\infty}^{\infty} n\, F_X^{n-1}(y)\, f_X(y)\, dy \qquad u = F_X(y) \qquad du = f_X(y)\, dy$$

$$= n \int_0^1 u^{n-1}\, du = \cancel{n} \frac{u^n}{\cancel{n}} \Big|_0^1 = 1^n - 0 = 1$$

7. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f(x) = e^{-x}$ for $x \geq 0$. Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_Y(y)$.

$$f_Y(y) = n F_X(y)^{n-1} f_X(y)$$

$$= \begin{cases} n(1-e^{-y})^{n-1} e^{-y} & \text{for } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

8. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f(x)$ and cumulative distribution function $F(x)$. Let $Y = \min(X_1, \ldots, X_n)$. Find the density $f_Y(y)$.

$$f_Y(y) = \frac{d}{dy} P(\min_i X_i \le y)$$

$$= \frac{d}{dy}\left(1 - P(\min_i X_i \ge y)\right)$$

$$= \frac{d}{dy}\left(1 - \prod_{i=1}^{n} P(X_i \ge y)\right)$$

$$= \frac{d}{dy}\left(1 - \prod_{i=1}^{n} (1 - F_X(y))\right)$$

$$= \frac{d}{dy}\left(1 - (1 - F_X(y))^n\right)$$

$$= (-1)n(1 - F_X(y))^{n-1}(-1) f_X(y)$$

$$= n(1 - F_X(y))^{n-1} f_X(y)$$
