Conditional Distributions and Independent Random Variables (Section 2.8)[1]

STA 256: Fall 2019

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Overview

Independent Random Variables: Discrete or Continuous

The real definition

The random variables X and Y are said to be *independent* if

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

For all subsets[2] A and B of the real numbers.

[2] Okay, all Borel subsets.

Big Theorem

We will use this as our criterion of independence

The random variables X and Y are independent if and only if

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

For all real x and y.

Theorem (for discrete random variables)

Recalling independence means $F_{X,Y}(x, y) = F_X(x)F_Y(y)$

The discrete random variables X and Y are independent if and only if

$$p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$$

for all real x and y.

Theorem (for continuous random variables)

Recalling independence means $F_{X,Y}(x, y) = F_X(x)F_Y(y)$

The continuous random variables X and Y are independent if and only if

$$f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$$

at all continuity points of the densities.

Conditional Distributions

Of discrete random variables

If X and Y are discrete random variables, the conditional probability mass function of X given $Y = y$ is just a conditional probability. It is given by

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

These are just probabilities of events. For example,

$$P(X = x, Y = y) = P\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}$$

We write

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

Note that $p_{X|Y}(x|y)$ is defined only for y values such that $p_Y(y) > 0$.

Conditional Probability Mass Functions

Both ways

$$p_{Y|X}(Y|X) = \frac{p_{X,Y}(x, y)}{p_X(x)}$$

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

Defined where the denominators are non-zero.

Independence makes sense

In terms of conditional probability mass functions

Suppose X and Y are independent. Then $p_{X,Y}(x,y) = p_X(x)p_Y(y)$, and

$$\begin{aligned} p_{X|Y}(x|y) &= \frac{p_{X,Y}(x,y)}{p_Y(y)} \\ &= \frac{p_X(x)p_Y(y)}{p_Y(y)} \\ &= p_X(x) \end{aligned}$$

So we see that the conditional distribution of X given $Y = y$ is identical for every value of y. It does not depend on the value of y.

The other way

Suppose the conditional distribution of X given $Y = y$ does not depend on the value of y. Then

$$
\begin{aligned}
& p_{X|Y}(x|y) = p_X(x) \\
\Leftrightarrow \quad & p_X(x) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \\
\Leftrightarrow \quad & p_{X,Y}(x,y) = p_X(x)\, p_Y(y)
\end{aligned}
$$

So that X and Y are independent.

Conditional distributions of continuous random variables

If X and Y are continuous random variables, the conditional probability density of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Note that $f_{X|Y}(x|y)$ is defined only for y values such that $f_Y(y) > 0$.
- It looks like we are conditioning on an event of probability zero, but the conditional density is a limit of a conditional probability, as the radius of a tiny region surrounding (x, y) goes to zero.

Conditional Probability Density Functions

Both ways

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Defined where the denominators are non-zero.

Independence makes sense

In terms of conditional densities

Suppose X and Y are independent. Then $f_{X,Y}(x,y) = f_X(x) f_Y(y)$, and

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{f_X(x) f_Y(y)}{f_Y(y)} \\ &= f_X(x) \end{aligned}$$

And we see that the conditional density of X given $Y = y$ is identical for every value of y. It does not depend on the value of y.

The other way

Suppose the conditional density of X given $Y = y$ does not depend on the value of y. Then

$$
\begin{aligned}
& f_{X|Y}(x|y) = f_X(x) \\
\Leftrightarrow \quad & f_X(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
\Leftrightarrow \quad & f_{X,Y}(x,y) = f_X(x)\, f_Y(y)
\end{aligned}
$$

So that X and Y are independent.

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