Conditional Probability and Independence[1] (Section 1.5)

STA 256: Fall 2019

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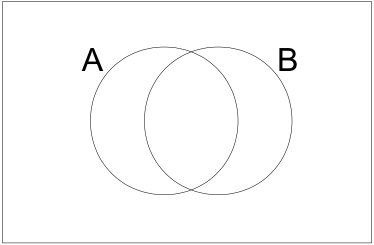
Overview

Conditional Probability: The idea

- If event A has occurred, maybe the probability of B is different from the probability of B overall.
- Maybe the chances of an auto insurance claim are different depending on the type of car.
- We will talk about the *conditional* probability of an insurance claim *given* that the car is a Dodge Charger.
- Or the conditional probability of graduating within five years, given that the student works full time during the school year.

Restrict the sample space

To condition on the event A, make A the new, restricted sample space.

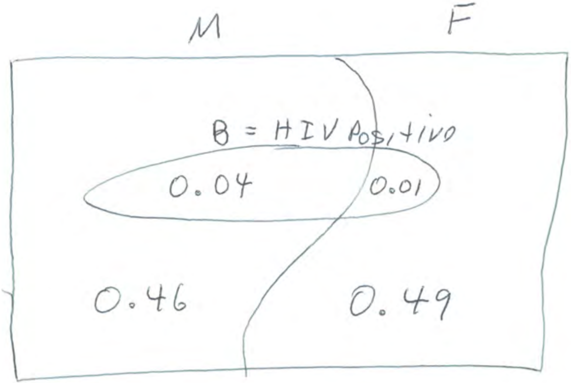


Express the probability of B as a fraction of the probability of A, provided the probability of A is not zero.

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Definition: The probability of B given A

$$\text{If } P(A) > 0,\ P(B|A) = \frac{P(A \cap B)}{P(A)}$$



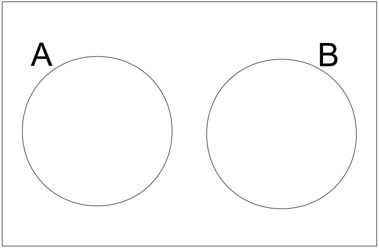
$$P(B|F) = \frac{P(A \cap F)}{P(F)} = \frac{0.01}{0.50} = 0.02$$

Either Direction

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Or,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



Multiplication Formula (1.5.2)

$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(A)P(B|A)$

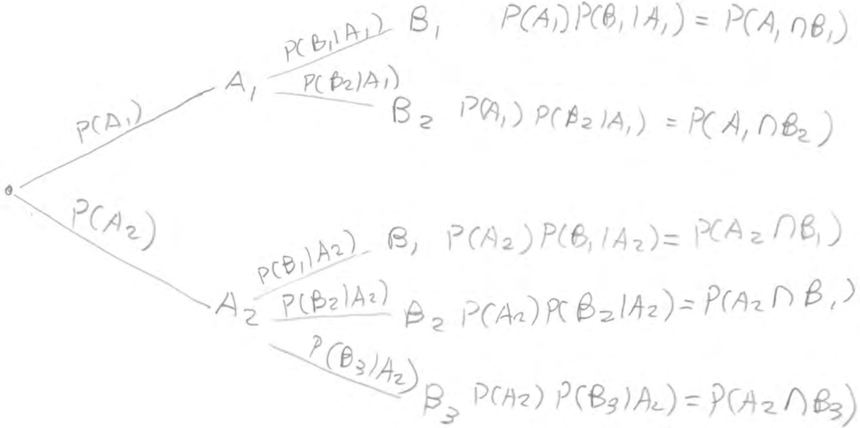
$$P(A \cap B) = P(A)P(B|A)$$

Useful for sequential experiments. A jar contains 15 red balls and 5 blue balls. What is the probability of randomly drawing a red and then a blue?

$P(R_1 \cap B_2) = P(R_1)P(B_2|R_1) = \frac{15}{20}\frac{5}{19} = \frac{15}{76} \approx 0.197$

Make a Tree

Justified by the multiplication formula: Not in the text



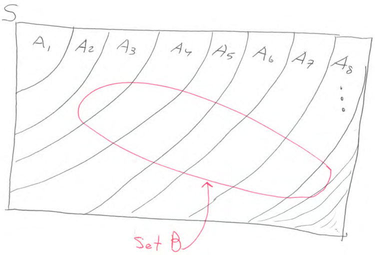
Trees

- Can be extended to more than 2 stages.
- Are best for *small* sequential experiments.
- Can allow you to side-step two important theorems, if the problem is set up nicely for you.
  - The Law of Total Probability
  - Bayes' Theorem.

Law of Total Probability, conditioned version

Theorem 1.5.1 in the text

Partition S into $A_1, A_2, \ldots$, disjoint, with $P(A_k) > 0$ for all k.



$B = \cup_{k=1}^{\infty}(A_k \cap B)$, disjoint

$$
\begin{aligned}
P(B) &= \sum_{k=1}^{\infty} P(A_k \cap B) \\
&= \sum_{k=1}^{\infty} P(A_k)P(B|A_k)
\end{aligned}
$$

Applies to finite partitions too.

Example

Law of Total Probability: $P(B) = \sum_{k=1}^{n} P(A_k)P(B|A_k)$

A jar contains two fair coins and one fair die. The coins have a "1" on one side and a "2" on the other side. Pick an object at random, roll or toss, and observe the number.

Can do this with a tree and get $P(2) = \frac{7}{18}$. Or,

$$\begin{aligned} P(2) &= P(\text{Coin 1})\,P(2|\text{Coin 1}) \\ &+ P(\text{Coin 2})\,P(2|\text{Coin 2}) \\ &+ P(\text{Die})\,P(2|\text{Die}) \\ &= \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{6} \\ &= \frac{7}{18} \end{aligned}$$

Thomas Bayes (1701-1761)

Image from the Wikipedia

Bayes Theorem: The idea

Bayes' Theorem allows you to turn conditional probability around, and obtain $P(A|B)$ from $P(B|A)$.

Bayes' Theorem

Our text gives the simplest version

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

Proof:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)P(B|A)}{P(B)} \end{aligned}$$

Two more versions of Bayes' Theorem

$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$ and use Law of Total Probability on $P(B)$

Let $S = \cup_{k=1}^{\infty} A_k$, disjoint, with $P(A_k) > 0$ for all k. Then

$$P(A_j|B) = \frac{P(A_j)P(B|A_j)}{\sum_{k=1}^{\infty} P(A_k)P(B|A_k)}$$

An important special case is

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

Independence: The idea

- Independent means totally unrelated.
- Occurrence of the event A tells you *nothing* about whether event B will occur.
- If we say that (for people without vanity plates) that the license plate number is independent of whether the car will get in an accident, it means that the license plate number has *no connection* to whether the car gets in an accident.
- If we say that taking Vitamin C supplements is independent of whether you get cancer, it means that taking Vitamin C supplements has *no connection* to whether you get cancer or not.
- It's a strong statement.
- It has a precise technical definition.

Independence of two events: Definition

Suppose $P(B|A) = P(B)$.

$$\begin{aligned} \frac{P(A \cap B)}{P(A)} &= P(B) \\ \Rightarrow \quad P(A \cap B) &= P(A)\, P(B) \end{aligned}$$

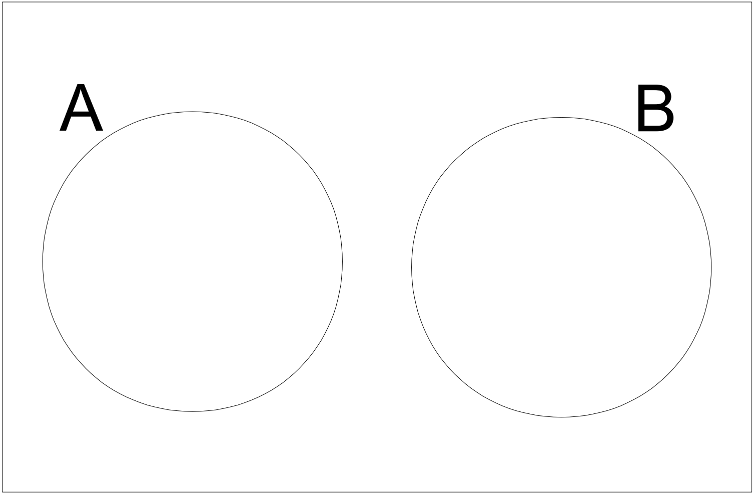
We use this *definition*. We say the events A and B are independent when

$$P(A \cap B) = P(A)\, P(B)$$

It's symmetric, and applies even if $P(A) = 0$ or $P(B) = 0$.

Independent is not the same as disjoint!

A very common error



Independence of several events

A set of events $A_1, \ldots, A_n$ are said to be *independent* if the probability of the intersection of any sub-collection is the product of probabilities.

Pairwise independence is not enough.

Example: Pairwise independent but not independent

A fair coin is tossed twice. Outcomes are HH, HT, TH, TT
Let

A = Head on first toss.

B = Head on second toss.

C = Exactly one Head.

$P(A) = P(B) = P(C) = \frac{1}{2}$
$P(A \cap B) = \frac{1}{4} = P(A)P(B)$
$P(A \cap C) = P(\mathrm{HT}) = \frac{1}{4} = P(A)P(C)$
And similarly for $P(B \cap C)$
But $P(A \cap B \cap C) = P(\emptyset) = 0 \neq \frac{1}{8}$.

“The dice have no memory”

Outcomes of simple statistical experiments like repeatedly flipping a coin or rolling a die will always be assumed independent.

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