STA 256f19 Assignment Nine[1]

Please read Section 3.4 in the text. Also, look over your lecture notes. The following homework problems are not to be handed in. They are preparation for Term Test 3 and the final exam. Use the formula sheet.

1. Let X have a moment-generating function $M_X(t)$ and let a be a constant. Show $M_{aX}(t) = M_X(at)$.

2. Let X have a moment-generating function $M_X(t)$ and let a be a constant. Show $M_{a+X}(t) = e^{at} M_X(t)$.

3. Let X_1 and X_2 be independent, discrete random variables, and let $Y = g(X_1) + h(X_2)$. Show $M_Y(t) = M_{g(X_1)}(t)\, M_{h(X_2)}(t)$. Because the random variables are discrete, you will add rather than integrating.

4. In the following table, derive the moment-generating functions (given on the formula sheet), and then use them to obtain the expected values and variances. To make the task shorter, notice that the Bernoulli is a special case of the binomial, and that the exponential and chi-squared distributions are special cases of the gamma. Chi-squared is a gamma with $\alpha = \nu/2$ and $\lambda = \frac{1}{2}$; exponential is a gamma with $\alpha = 1$. Do the general cases first and then just write the answer for the special cases.

Distribution	**MGF** $M_x(t)$	$E(X)$	$Var(X)$
Bernoulli (θ)			
Binomial (n, θ)			
Poisson (λ)			
Exponential (λ)			
Gamma (α, λ)			
Normal (μ, σ^2)			
Chi-squared (ν)			

5. Let X be a geometric random variable with parameter θ.
   (a) Find the moment-generating function.
   (b) Differentiate to obtain $E(X)$.

6. Let $X \sim N(\mu, \sigma^2)$. Show $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$ using moment-generating functions.

7. Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be independent. Find the distribution of $Y = X_1 + 3X_2$.

8. Let $X_1, \ldots, X_n$ be independent Bernoulli(θ) random variables. Find the distribution of $Y = \sum_{i=1}^n X_i$.

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9. Let $X_1, \dots, X_n$ be independent Normal(μ, σ^2) random variables. Find the distribution of the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

10. Let $X_1, \dots, X_n$ be independent Gamma(α, λ) random variables. Find the distribution of the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

11. Let $Z \sim N(0,1)$ and let $Y = Z^2$. Find the distribution of Y using moment-generating functions.

12. Let X be a *degenerate* random variable with $P(X = \mu) = 1$.

    (a) Find the moment-generating function.

    (b) Differentiate to obtain $E(X)$ and $Var(X)$. Do these answers make sense?

    (c) Comparing this to the moment-generating function of a normal, one can say that in a weird way, a degenerate distribution is normal with variance _____.

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13. Suppose that X and Y are discrete independent random variables with the following moment generating functions:

$$M_X(t) = E(e^{tX}) = \frac{1}{6}e^{t} + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$$

$$M_Y(t) = E(e^{tY}) = \frac{1}{10}e^{-t} + \frac{4}{10}e^{2t} + \frac{1}{2}e^{3t}.$$

    Using the moment generating function, find the distribution of

    (a) $Z = X + Y$.

    (b) $U = X - Y$.

14. Let X and Y be discrete random variables such that

$$p_X(x) = \frac{1}{3}, \quad x = -1, 0, 1$$

    and

$$p_Y(y) = \frac{1}{2}, \quad y = 2, 4.$$

    Let $Z = X + Y$.

    (a) Using the probability mass functions of X and Y, find the probability mass function of Z.

    (b) Find the moment generating function of Z.

    (c) Using part (b), find the probability mass function of Z. Does your answer agree with (a)?

15. Let X and Y be independent random variables, both with $Poisson(\lambda)$ distribution, for some $\lambda > 0$. Define $Z = X + Y$.

    (a) Find the distribution of Z by using the moment generating function.

    (b) For any non-negative integer n, find the conditional probability mass function of X given $Z = n$.

(c) State the name of the conditional distribution of X given $Z = n$.

16. Let X be a continuous random variable with pdf $f(x) = ke^{-|x|}, -\infty < x < \infty$.

    (a) Find the value of the constant k.
    (b) Find the moment generating function of X.
    (c) Find the mean and the variance of X.

17. Let $M_X(t)$ be the moment generating function of a random variable X.

    (a) Show that $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} E(X^k)\frac{t^k}{k!}$. **Hint:** $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
    (b) It follows form part (a) that, in Maclaurin (power) series, $E(X^k)$ is the coefficient of $t^k/k!$, $k = 0, 1, 2, \ldots$. Use this fact to find $E(X^{1024})$ in the following cases:
        i. $M_X(t) = \frac{1}{1-t^2}$, $|t| < 1$.
        ii. $M_X(t) = e^{t^2}$.
        iii. $M_X(t) = \frac{1}{(1-5t)^2}$, $|5t| < 1$.

18. Let $M_X(t)$ be the moment generating function of a random variable X. Define $S(t) = \log M_X(t)$.

    (a) Find $S(0)$
    (b) Show that $\frac{d}{dt}S(t)|_{t=0} = E(X)$.
    (c) Show that $\frac{d^2}{dt^2}S(t)|_{t=0} = Var(X)$.

19. Let $X \sim N(\mu = 1, \sigma^2 = 4)$. If $Y = 0.5^X$, find $E(Y^2)$. **Hint:** Use moment generating function.

20. If $M_X(t) = e^{3t+8t^2}$ is the moment generating function of the random variable X, find $P(-1 < X < 9)$.
