STA 256 F 19 Assignment 8

(1) 3.1.1(a) $E(X) = (-4)\cdot\frac{1}{7} + (0)\cdot\frac{2}{7} + (3)\cdot\frac{4}{7}$

$$= \frac{-4}{7} + \frac{12}{7} = 8/7$$

(2) Show if $P(X \geq 0) = 1$ then $E(X) \geq 0$.

(a) Discrete For $x \in A$ with $P(X \in A) = 1$,

$$x \geq 0 \Rightarrow x\, P_X(x) \geq 0 \cdot P_X(x) = 0$$

$$\Rightarrow \underbrace{\sum_{x \in A} x\, P_X(x)}_{= E(X)} \geq \sum_{x \in A} 0 = 0$$

(b) Continuous

$$x \geq 0 \Rightarrow x f_X(x) \geq 0 \cdot f_X(x) = 0$$

$$\Rightarrow \underbrace{\int_0^\infty x f_X(x)\, dx}_{= E(X)} \geq \int_0^\infty 0\, dx = 0$$

(3) $E(aX) = \int_{-\infty}^{\infty} ax\, f_X(x)\, dx = a\int_{-\infty}^{\infty} x f_X(x)\, dx$

$$= a\, E(X)$$

(4) $$E(X+Y) = \sum_x \sum_y (x+y)\, P_{XY}(x,y)$$

$$= \sum_x \sum_y x\, P_{XY}(x,y) + \sum_x \sum_y y\, P_{XY}(x,y)$$

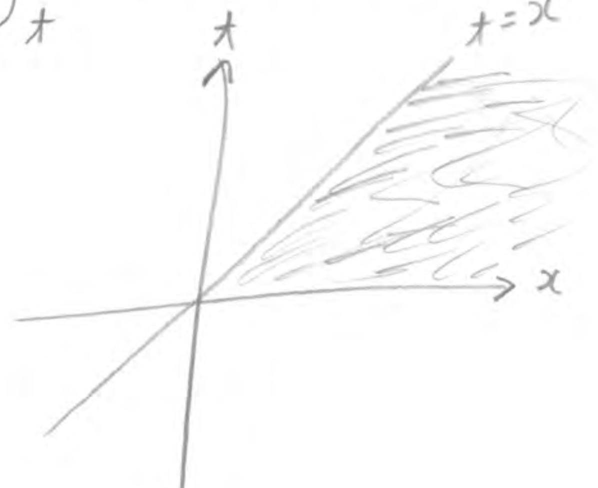
$$= E(X) + E(Y)$$

(5) $$E[E(Y|X)] = \sum_x E(Y|x)\, p_X(x)$$

$$= \sum_x \left(\sum_y y\, P_{Y|X}(y|x) \right) P_X(x)$$

$$= \sum_x \sum_y y\, \frac{P_{XY}(x,y)}{\cancel{P_X(x)}}\, \cancel{P_X(x)} = E(Y)$$

(6) $$\int_0^\infty P(X>t)\, dt = \int_0^\infty \int_t^\infty f_X(x)\, dx\, dt$$



$$= \int_0^\infty \int_0^x f_X(x)\, dt\, dx$$

$$= \int_0^\infty f_X(x) \left(\int_0^x dt \right) dx$$

$$= \int_0^\infty x\, f_X(x)\, dx = E(X)$$

⑦ Show $Var(a+X) = Var(X)$

Denoting $E(X)$ by μ_x, $E(a+X) = E(a)+E(X)$
$= a + \mu_X$

$$Var(a+X) = E\{(a+X - E(a+X))^2\}$$
$$= E\{(\cancel{a}+X-\cancel{a}-\mu_x)^2\}$$
$$= E\{(X-\mu_x)^2\} = Var(X)$$

⑧ $$Var(bX) = E\{(bX - E(bX))^2\}$$
$$= E\{(b(X-\mu_x))^2\} = E\{b^2(X-\mu_x)^2\}$$
$$= b^2 E\{(X-\mu_x)^2\} = b^2 Var(X)$$

⑨ $$Var(X) \overset{def}{=} E\{(X-\mu_x)^2\}$$
$$= E(X^2 - 2X\mu_x + \mu_x^2)$$
$$= E(X^2) - 2\mu_x E(X) + E(\mu_x^2)$$
$$= E(X^2) - 2\mu_x^2 + \mu_x^2 = E(X^2) - \mu_x^2$$
$$= E(X^2) - (E[X])^2$$

(10)
(a) $E(X^k) = \int_0^1 x^k \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}} \frac{\Gamma(\alpha+k)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+k)} \underbrace{\int_0^1 \frac{\Gamma(\alpha+k+\beta)}{\Gamma(\alpha+k)\Gamma(\beta)} x^{\alpha+k-1}(1-x)^{\beta-1}\,dx}_{=1}$$

$$= \frac{\Gamma(\alpha+k)\ \Gamma(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\alpha+\beta+k)}$$

(b) $\alpha = \beta = 1$

(c) $Var(X) = E(X^2) - (E(X))^2$

$$E(X) = \frac{\Gamma(1+1)\Gamma(1+1)}{\Gamma(1)\Gamma(1+1+1)} = \frac{1!\,1!}{0!\,2!} = \frac{1}{2}$$

$E(X^2)$ with $k=2$ is

$$\frac{\Gamma(1+2)\Gamma(2)}{\Gamma(1)\ \Gamma(4)} = \frac{2!}{3!} = \frac{1}{3}, \text{ so}$$

$$Var(X) = E(X^2) - (EX)^2 = \frac{1}{3} - \frac{1}{4}$$

$$= \frac{4}{12} - \frac{3}{12} = 1/12$$

e

(11) This is the Petersburg Paradox described on p. 133 of the text. No admission fee is great enough.

(12) (a) Bernoulli: Ex. 3.1.6 on p. 131

(b) Binomial: Ex. 3.1.7 on p. 131

(c) Geometric: Ex. 3.1.8 on p. 132 pretty clever

(d) Poisson: Ex. 3.1.9 on p. 132

(e) Gamma: $E(X^k) = \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)\lambda^k}$, Sample Question 6 for expected value (Lecture Section LEC0101)

(f) Beta: Problem 10 of this problem set.

(13) IS ON THE NEXT PAGE.

(14) 3.1.5 with $E(X) = \frac{1-\theta}{\theta}$ and $E(Y) = \lambda$

$E(8X - Y + 12) = 8E(X) - E(Y) + E(12)$

$= \frac{8(1-\theta)}{\theta} - \lambda + 12$

3.1.7 By independence,

$E(XY) = E(X)E(Y)$

$= \left(\frac{80}{4}\right) \cdot \frac{3}{2} = \frac{20 \cdot 3}{2} = 30$

3.1.9 $E(X) = (8-4) \cdot \frac{1}{2} + 8 \cdot \frac{1}{2} = \frac{4+8}{2}$

$= 6$

f

13 $E(XY) = \sum_x \sum_y xy\, P_{XY}(x,y)$

$$\overset{\text{ind}}{=} \sum_x \sum_y xy\, P_X(x) P_Y(y)$$

$$= \sum_x x\, P_X(x) \sum_y y\, P_Y(y) = E(X)E(Y)$$

14 IS ON THE PRECEDING PAGE

15 3.1.11 Roll a die: Expected # is 3.5 (balance point)

(a) $E(X+Y) = E(X)+E(Y) = 2(3.5) = 7$

(b) $E(XY) \overset{\text{ind}}{=} E(X)E(Y) = (3.5)(3.5) = 12.25$

16 3.1.13

Letting X = the number of coins tossed
Y = Total number of heads

$$E(Y) = E[E(Y|X)] = \sum_{n=1}^{6} E(Y|X=n)\cdot\frac{1}{6}$$

$$= \sum_{n=1}^{6} \underbrace{n\cdot\tfrac{1}{2}}_{\text{Binomial Expected value}}\cdot\frac{1}{6} = \frac{1}{12}\sum_{n=1}^{6} n$$

$$= \frac{1}{12}\cdot\frac{6(6+1)}{2} = \frac{21}{12} = \frac{7}{4}$$

17 3.2.1c

$$1 = \int_{-5}^{-2} c x^4 \, dx = c \frac{x^5}{5} \Big|_{-5}^{-2} = \frac{c}{5}\left((-2)^5 - (-5)^5\right)$$

$$= \frac{c}{5}(-32 - -3125) = \frac{c}{5}(3093)$$

$$\Rightarrow c = \frac{5}{3093}, \text{ and}$$

$$E(X) = \frac{5}{3093} \int_{-5}^{-2} x^5 \, dx = \frac{5}{3093} \frac{x^6}{6} \Big|_{-5}^{-2}$$

$$= \frac{5}{3093 \cdot 6}\left(2^6 - 5^6\right) = \frac{5(-15561)}{18558}$$

$$= -\frac{5 \cdot 15561}{6 \cdot 3 \cdot 1031} = -\frac{5 \cdot \cancel{3} \cdot \cancel{3} \cdot 7 \cdot 13 \cdot 19}{\cancel{3} \cdot 2 \cdot \cancel{3} \cdot 1031}$$

$$= -\frac{8645}{2062} = -4.192532$$

18

3.2.3a

$$E(X) = \int_0^1 \int_0^3 x \, \frac{4xy + 3x^2y^2}{18} \, dy \, dx$$

$$= \frac{1}{18} \int_0^1 \left(\int_0^3 (4x^2y + 3x^3y^2) \, dy \right) dx$$

$$= \frac{1}{18} \int_0^1 4x^2 \frac{y^2}{2} \Big|_0^3 + \cancel{3}x^3 \frac{y^3}{\cancel{3}} \Big|_0^3 \, dx$$

$$= \frac{1}{18} \int_0^1 (18x^2 + 27x^3) \, dx = \frac{1}{18} \left(18 \frac{x^3}{3} \Big|_0^1 + 27 \frac{x^4}{4} \Big|_0^1 \right.$$

$$= \frac{1}{18} \left(6 + \frac{27}{4} \right) = \frac{1}{18} \left(\frac{24+27}{4} \right) = \frac{51}{3 \cdot 3 \cdot 2 \cdot 2 \cdot 2}$$

$$= \frac{\cancel{3} \cdot 17}{\cancel{3} \cdot 3 \cdot 8} = \frac{17}{24} \quad \checkmark$$

18 continued

3.2.3 f

$$E(X^2Y^3) = \int_0^1 \int_0^3 x^2 y^3 \, \frac{4xy + 3x^2y^2}{18} \, dy \, dx$$

$$= \frac{1}{18} \int_0^1 \int_0^3 (4x^3y^4 + 3x^4y^5) \, dy \, dx$$

$$= \frac{1}{18} \int_0^1 \left(4x^3 \frac{y^5}{5} \Big|_0^3 + 3x^4 \frac{y^6}{6} \Big|_0^3 \right) dx$$

$$= \frac{1}{18} \int_0^1 \left(\frac{4}{5} \cdot 3^5 \, x^3 + \frac{3^6}{2} x^4 \right) dx$$

$$= \frac{1}{18} \left(\frac{\cancel{4} \cdot 3^5}{5} \frac{x^4}{\cancel{4}} \Big|_0^1 + \frac{3^6}{2} \frac{x^5}{5} \Big|_0^1 \right)$$

$$= \frac{1}{18} \left(\frac{2 \cdot 3^5}{2 \cdot 5} + \frac{3^6}{10} \right) = \frac{3^5}{3 \cdot 3 \cdot 2} \left(\frac{2}{10} + \frac{3}{10} \right)$$

$$= \frac{3^3}{2} \left(\frac{5}{10} \right) = \frac{3^3}{4} = \frac{27}{4}$$ ✓

k

(19) 3.2.11 $E(X+Y) = E(X) + E(Y)$
$= 174 + 160 = 334$

(20) (a) $\mathrm{Cov}(X,Y) = E\{(X-\mu_x)(Y-\mu_y)\}$

$= E(XY - X\mu_y - \mu_x Y + \mu_x \mu_y)$

$= E(XY) - \mu_x \mu_y - \mu_x \mu_y + \mu_x \mu_y)$

$= E(XY) - \mu_x \mu_y = E(XY) - E(X)E(Y)$

(b) By (20a), $\mathrm{Cov}(XY) = E(XY) - E(X)E(Y)$, which equals 0 by problem (13) if X & Y are independent.

(c) $\mathrm{Cov}(a+X, b+Y) = E\{(a+X-E(a+X))(b+Y-E(b+Y))\}$

$= E\{(\cancel{a}+X\cancel{-a}-E(X))(\cancel{b}+Y\cancel{-b}-E(Y))\}$

$= E\{(X-E(X))(Y-E(Y))\} = \mathrm{Cov}(X,Y)$

(d) $\mathrm{Cov}(aX, bY) = E\{(aX - E(aX))(bY - E(bY))\}$

$= E\{(aX - aE(X))(bY - bE(Y)\}$

$= E\{a(X-E(X))b(Y-E(Y))\}$

$= ab\,E\{(X-E(X))(Y-E(Y))\} = ab\,\mathrm{Cov}(X,Y)$

20e $Cov(X, Y+Z) = E\{(X-E(X))(Y+Z-E(Y+Z))\}$

$= E\{(X-E(X))((Y-E(Y)) + (Z-E(Z)))\}$

$= E\{(X-E(X))(Y-E(Y))\} + E\{(X-E(X))(Z-E(Z))\}$

$= Cov(X,Y) + Cov(X,Z)$

(f) $Var(aX+bY) = E\{(aX+bY - E(aX+bY))^2\}$

$= E(aX+bY - aE(X) - bE(Y))^2$

$= E(a(X-E(X)) + b(Y-E(Y)))^2$

$= E(a^2(X-E(X))^2 + 2a(X-E(X))b(Y-E(Y)) + b^2(Y-E(Y))^2)$

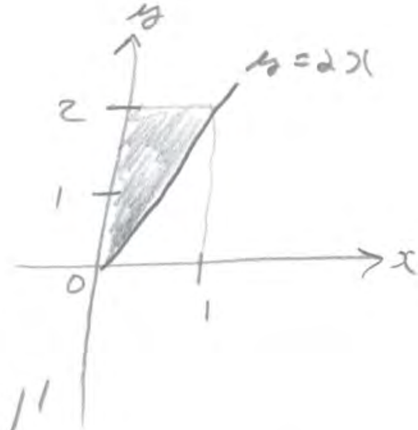
$= a^2E(X-E(X))^2 + 2ab\,E(X-E(X))(Y-E(Y)) + b^2E(Y-E(Y))^2$

$= a^2 Var(X) + b^2 Var(Y) + 2ab\, cov(X,Y)$

m

(21) $f_{XY}(x,y) = \begin{cases} 1 & \text{for } \begin{array}{l} 0 \le x \le 1 \\ 2x \le y \le 2 \end{array} \\ 0 & \text{otherwise} \end{cases}$

(a) For cov(X,Y), need



$f_X(x) = \int_{2x}^{2} dy = 2 - 2x = 2(1-x)$, for $0 \le x \le 1$

So $E(X) = \int_0^1 x\, 2(1-x)\, dx$

$= 2\int_0^1 (x - x^2)\, dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2\left(\frac{3}{6} - \frac{2}{6}\right) = 2\left(\frac{1}{6}\right) = \frac{1}{3}$

And for $0 \le y \le 2$,

$f_Y(y) = \int_0^{y/2} 1\, dx = \frac{y}{2}$, so

$E(Y) = \int_0^2 y \frac{y}{2}\, dy = \frac{1}{2}\frac{y^3}{3}\Big|_0^2 = \frac{1}{6}\cdot 8$

$= \frac{4}{3}$, and

$E(XY) = \int_0^1 \int_{2x}^2 xy\, dy\, dx = \int_0^1 x \int_{2x}^2 y\, dy\, dx$

$= \int_0^1 x \frac{y^2}{2}\Big|_{2x}^2 dx = \frac{1}{2}\int_0^1 x(2^2 - 4x^2)\, dx$

$= \frac{1}{2}\int_0^1 x(4 - 4x^2)\, dx = \frac{4}{2}\int_0^1 (x - x^3)\, dx$

$= 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = 2\left(\frac{2-1}{4}\right) = \frac{1}{2}$, so

$cov(X,Y) = EXY - EXEY = \frac{1}{2} - \frac{1}{3}\cdot\frac{4}{3} = \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$

11

21b) To get $E(X \mid Y=y)$, need

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{y/2}$$

$$= \frac{2}{y} \text{ for } 0 \le x \le y/2 \quad \text{Uniform}\left(0, \theta = \frac{y}{2}\right)$$

So that $E(X \mid Y=y)$ should be $\frac{1}{2}\frac{y}{2} = \frac{y}{4}$, and

$$\int_0^{y/2} x \cdot \frac{2}{y}\, dx = \frac{\cancel{2}}{y} \frac{x^2}{\cancel{2}} \Big|_0^{y/2} = \frac{1}{y} \frac{y^2}{2^2} = \frac{y}{4}$$

This only makes sense for $0 \le y \le 2$, because that's when $f_Y(y) \ne 0$.

c) No, because $f_X(x) f_Y(y) = 2(1-x) \cdot \frac{y}{\cancel{2}}$
$= y(1-x) \ne 1$ where $f_{XY}(x,y) \ne 0$

d) $E(X) = E[E(X \mid Y)] = \int_0^2 \underbrace{\frac{y}{4}}_{E(X|Y=y)} \cdot \underbrace{\frac{y}{2}}_{f_Y(y)}\, dy$

$$= \int_0^2 \frac{y^2}{8}\, dy$$

$$= \frac{1}{8} \frac{y^3}{3}\Big|_0^2 = \frac{1}{8}\cdot\frac{1}{3}(8-0) = \frac{1}{3},$$

which we also got as part of (a)

(22)

	$x=1$	$x=2$	$x=3$	
$y=1$	$3/12$	$1/12$	$3/12$	$\frac{7}{12}$
$y=2$	$1/12$	$3/12$	$\frac{1}{12}$	$\frac{5}{12}$
	$\frac{4}{12}$	$\frac{4}{12}$	$\frac{4}{12}$	

(a) $E(X|Y=1) = \sum_{x=1}^{3} x\, P_{X|Y}(x|1) = \sum_{x=1}^{3} x \frac{P_{XY}(x,y)}{7/12}$

$= 1\cdot\frac{3}{7} + 2\cdot\frac{1}{7} + 3\cdot\frac{3}{7} = \frac{1}{7}(3+2+9)$

$= \frac{14}{7} = 2$ Balance point

(b) $E(Y^2|X=2) = 1^2\cdot\frac{1}{4} + 2^2\cdot\frac{3}{4} = \frac{1}{4} + \frac{12}{4} = \frac{13}{4}$

(c) $Cov(X,Y) = EXY - EXEY$

$E(XY) = 1\cdot\frac{3}{12} + 2\cdot\frac{1}{12} + 3\cdot\frac{3}{12}$
$\quad + 2\cdot\frac{1}{12} + 4\cdot\frac{3}{12} + 6\cdot\frac{1}{12} = \frac{34}{12} = \frac{17}{6}$

$E(X) = 2 = \frac{24}{12}$, $E(Y) = 1\cdot\frac{7}{12} + 2\cdot\frac{5}{12} = \frac{17}{12}$

so $Cov(X,Y) = \frac{34\cdot 12}{12\cdot 12} - \frac{24}{12}\cdot\frac{17}{12} = \frac{(408-408)}{144}$

$= 0$

(d) No, not independent, since

$P(X=1, Y=1) = \frac{1}{4} \neq P(X=1)P(Y=1) = \frac{7}{12}\cdot\frac{4}{12}$

$= \frac{7}{36} \approx 0.1944$

P

(23)

	$x=0$	$x=1$
$y=0$	0 $P(X=0, Y=0)$	0 $P(X=1, Y=0)$
$y=1$	0 $P(X=0, Y=1)$	1 $P(X=1, Y=1)$

xy is written in red

So $E(XY) = P(X=1, Y=1)$. Then

$Cov(X,Y) = 0 \Rightarrow E(XY) = E(X)E(Y)$

$\Rightarrow P(X=1, Y=1) = \theta\psi = P(X=1)P(Y=1)$.

Then by Exercise 2.8.10, X & Y are independent. ■

(24) (a) $Cov(X_1 + X_2, X_1 - X_2)$ by an easy extension of Problem 20e

$= Cov(X_1, X_1) - \cancel{Cov(X_1, X_2)} + \cancel{Cov(X_2, X_1)}$

$- Cov(X_2, X_2) = Var(X_1) - Var(X_2)$

(b) will $= 0$ if $Var(X_1) = Var(X_2)$

(25) By Problem 20 parts (c) and (d)

$Cov(a+bX, c+dY) = cd\, Cov(X,Y)$

By Problems 7 and 8 $Var(a+bX) = b^2 Var(X)$ and $Var(c+dY) = d^2 Var(Y)$, so

$$\frac{Cov(a+bX, c+dY)}{\sqrt{Var(a+bX)Var(c+dY)}} = \frac{bd\, Cov(X,Y)}{\sqrt{b^2 d^2 Var(X)\, Var(Y)}}$$

$= corr(X,Y)$

26 (a) $\text{Cov}(X_j, \bar{X}) = \text{Cov}(X_j, \frac{1}{n}\sum_{i=1}^{n} X_i)$

$= \frac{1}{n}\text{Cov}(X_j, \sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n}\text{Cov}(X_i, X_j)$

$= \frac{1}{n}\left(\text{Cov}(X_j, X_j) + \underbrace{\sum_{i \neq j}\text{Cov}(X_i, X_j)}_{=0}\right)$

$= \frac{1}{n}\text{Var}(X_j) = \frac{\sigma_x^2}{n}$

(b) $\text{Cov}(\bar{X}, \bar{Y}) = \frac{1}{n^2}\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right)$

$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\text{Cov}(X_i, Y_i) + \sum\sum_{i \neq j}\text{Cov}(X_i, Y_j)\right)$

$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\sigma_{xy} + 0\right) = \frac{1}{n^2} n \sigma_{xy}$

$= \frac{\sigma_{xy}}{n}$

(c) $\text{Var}(\bar{X}) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\text{Var}\sum_{i=1}^{n} X_i$

$\overset{\text{ind}}{=} \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_x^2 = \frac{\sigma_x^2}{n}$, and similarly,

$\text{var}(\bar{Y}) = \frac{\sigma_y^2}{n}$, so

$\text{Corr}(\bar{X}, \bar{Y}) = \frac{\text{Cov}(\bar{X}, \bar{Y})}{\sqrt{\text{var}(\bar{X})\,\text{var}(\bar{Y})}} = \frac{\sigma_{xy}/n}{\sqrt{\sigma_x^2/n \; \sigma_y^2/n}}$

$= \frac{\cancel{\frac{1}{n}}\sigma_{xy}}{\cancel{\frac{1}{n}}\sqrt{\sigma_x^2 \sigma_y^2}} = \text{Corr}(X_i, Y_i)$

Note $\text{Var}(\bar{X})$ is buried in this problem.