STA 256f19 Assignment Eight[1]

Please read Section 3.1-3.3 in the text. Also, look over your lecture notes. The following homework problems are not to be handed in. They are preparation for Term Test 3 and the final exam. Use the formula sheet.

For some of these questions, it may not be clear whether you are supposed to use linear properties of expected value, or whether you are supposed to go back to the definition and use integration or summation. The rule is that if integration or summation is not explicitly mentioned, just use expected value signs.

1. Do Exercise 3.1.1 part (a) only.
2. Show that if $P(X \geq 0) = 1$, then $E(X) \geq 0$. Treat the discrete and continuous cases separately.
3. Let a be a constant, and let X be a continuous random variable, so you will integrate rather than adding. Show $E(aX) = a\,E(X)$.
4. Let X and Y be discrete random variables, so you will add rather than integrating. Show that $E(X+Y) = E(X)+E(Y)$. You are proving a special case of $E\left(\sum_{i=1}^n a_iX_i\right) = \sum_{i=1}^n a_iE(X_i)$ from the formula sheet, so you can't use that. If you assume independence, you get a zero.
5. Let X and Y be discrete random variables, so you will add rather than integrating. Prove the double expectation formula: $E(Y) = E[E(Y|X)]$.
6. Let X be a continuous random variable with $P(X \geq 0) = 1$. Prove $E(X) = \int_0^\infty P(X > t)\,dt$. Hint: Write the probability as an integral, sketch the region of integration, and switch order of integration using Fubini's Theorem.
7. Show $Var(a + X) = Var(X)$.
8. Show $Var(bX) = b^2Var(X)$
9. Show $Var(X) = E(X^2) - [E(X)]^2$
10. Let X have a beta distribution with parameters α and β.
    (a) Calculate $E(X^k)$.
    (b) The Uniform(0,1) distribution is a special case of the beta distribution. What are the parameters α and β?
    (c) Use your answer to Question 10a to show that the variance of the Uniform(0,1) distribution is 1/12.
11. To play this casino game, you must pay an admission fee. You toss a fair coin, and wait until the first head appears. Your payoff is 2^z pennies, where z is the number of tails that occur *before* the first head. What should the admission fee be in order to make this a "fair" game — that is, a game with expected value zero?

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12. Derive the expected value formulas for the following distributions. Most of this is in the text or lecture.

    (a) Bernoulli (θ): $E(X) = \theta$
    (b) Binomial (n, θ): $E(X) = n\theta$
    (c) Geometric (θ): $E(X) = \frac{1-\theta}{\theta}$
    (d) Poisson (λ): $E(X) = \lambda$
    (e) Gamma (α, λ): $E(X) = \alpha/\lambda$
    (f) Beta (α, β): $E(X) = \frac{\alpha}{\alpha+\beta}$

13. Let X and Y be independent (discrete) random variables, so you will add rather than integrating. Show $E(XY) = E(X)E(Y)$. This formula also holds for continuous random variables.

14. Do Exercises 3.1.5, 3.1.7 and 3.1.9.

15. Do Exercise 3.1.11 the easy way, using independence.

16. Do Exercise 3.1.13. Use double expectation.

17. Do Exercise 3.2.1 part (c) only.

18. Do Exercise 3.2.3 parts (a) and (f) only.

19. Do Exercise 3.2.11 the easy way. Is there any need to assume that people get married at random?

20. Prove the following facts about covariance.

    (a) $Cov(X, Y) = E(XY) - E(X)E(Y)$
    (b) If X and Y are independent, $Cov(X, Y) = 0$
    (c) $Cov(a + X, b + Y) = Cov(X, Y)$
    (d) $Cov(aX, bY) = abCov(X, Y)$
    (e) $Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)$
    (f) $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$

21. The continuous random variables X and Y have joint density function $f_{xy}(x, y) = 1$ for $0 < x < 1$ and $2x < y < 2$, and zero otherwise.

    (a) Find $cov(X, Y)$. My answer is $\frac{1}{18}$.
    (b) Find $E(X|Y = y)$ My answer is $\frac{y}{4}$, defined for $0 \leq y \leq 2$.
    (c) Are X and Y independent?
    (d) Find $E(X)$ using double expectation: $E(X) = E(E[X|Y])$. Compare to the $E(X) = \frac{1}{3}$ you get using the definition of expected value.

22. The discrete random variables X and Y have joint distribution

	$x = 1$	$x = 2$	$x = 3$
$y = 1$	3/12	1/12	3/12
$y = 2$	1/12	3/12	1/12

   (a) What is $E(X|Y = 1)$? [2]
   (b) What is $E(Y^2|X = 2)$? [13/3]
   (c) What is $Cov(X, Y)$? [0]
   (d) Are X and Y independent? Answer Yes or [No] and prove your answer.

23. The last question should persuade you that two random variables can have zero covariance without being independent. However, consider Exercise 2.8.10 on page 107 (an example that's surprisingly important). In that problem, $X \sim$ Bernoulli(θ) and $Y \sim$ Bernoulli(ψ), and you showed that if $P(X = 1, Y = 1) = P(X = 1)P(Y = 1)$, then X and Y were independent — something that is not true of random variables in general.

    For that same example, show that if $Cov(X, Y) = 0$, then X and Y are independent.

24. Let X_1 and X_2 be independent random variables; let $Y_1 = X_1 + X_2$, and $Y_2 = X_1 - X_2$.
   (a) Derive a general formula for $Cov(Y_1, Y_2)$.
   (b) Give a sufficient condition for $Cov(Y_1, Y_2) = 0$.

25. The correlation between two random variables X and Y is defined as

$$Corr(X, Y) = \rho = \frac{Cov(X, Y)}{\sqrt{Var(X)\,Var(Y)}}$$

    Find a formula for $Corr(a + bX, c + dY)$ in terms of $Corr(X, Y)$. Assume b and d are both non-zero. Show your work.

26. Let the pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ be selected independently from a joint distribution with $E(X_i) = \mu_x$, $E(Y_i) = \mu_y$, $Var(X_i) = \sigma^2_x$, $Var(Y_i) = \sigma^2_y$, and $Cov(X_i, Y_i) = \sigma_{xy}$. Independence means that X_i and Y_i are independent of X_j and Y_j for $i \neq j$.

    The sample mean (average) of the X values is $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$, and the sample mean of the Y values is $\overline{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$
   (a) Show $Cov(X_j, \overline{X}) = \sigma^2_x/n$.
   (b) Show $Cov(\overline{X}, \overline{Y}) = \sigma^2_{xy}/n$
   (c) Show $Corr(\overline{X}, \overline{Y}) = Corr(X_i, Y_i)$.
