STA 256f19 Assignment Seven[1]

Please read Sections 2.8 and 2.9 in the text. Note that in a departure from the text, the criterion for independence is in this class is $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for all real x and y. Also, look over your lecture notes. The following homework problems are not to be handed in. They are preparation for Term Test 3 and the final exam.

1. Let X and Y be continuous random variables.
   (a) Prove that if $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ for all real x and y, then the random variables X and Y are independent. This result is also true if the condition holds except on a set of probability zero.
   (b) Prove that if X and Y are independent, then $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ at all points where $F_{xy}(x,y)$ is differentiable and $f_{xy}(x,y)$ is continuous.
2. Let X and Y be discrete random variables. Prove that if $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$, then X and Y are independent.
3. Do exercises 2.8.1, 2.8.3, 2.8.5 in the text.
4. Do exercise 2.8.8 in the text. The answer is 2/5. To make this problem easier, first prove that $P(Y > 5) = \int_{-\infty}^{\infty} P(Y > 5|X = x) f_X(x)\, dx$. Write that conditional probability as an integral with repect to a conditional density, and switch order of integration.
5. Let $p_{X,Y}(x,y) = \frac{xy}{36}$ for $x = 1, 2, 3$ and $y = 1, 2, 3$, and zero otherwise.
   (a) What is $p_{Y|X}(1|2)$? Answer is 1/6.
   (b) What is $p_{X|Y}(1|2)$? Answer is 1/6.
   (c) Are X and Y independent? Answer Yes or No and prove your answer.
6. The continuous random variables X and Y have joint probability density function
$$f_{X,Y}(x,y) = \begin{cases} k\, x^2 y & \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x^2 \\ 0 & \text{otherwise} \end{cases}$$
   (a) What is the value of k? Answer is $k = 14$.
   (b) Find the marginal density function $f_X(x)$. Do not forget to indicate where the density is non-zero.
   (c) Find the marginal density function $f_Y(y)$. Do not forget to indicate where the density is non-zero.
   (d) Find the conditional density function $f_{X|Y}(x|y)$. Do not forget to indicate where the density is non-zero. For what values of y is this conditional density defined?
   (e) Find the conditional density function $f_{Y|X}(y|x)$. Do not forget to indicate where the density is non-zero. For what values of x is this conditional density defined?
   (f) Are X and Y independent? Answer Yes or No and justify your answer.

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7. This question is from the 2018 final exam. There is a continuous version of Bayes' Theorem, which says

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|t)f_Y(t)\,dt}$$

Prove it. It's helpful to start with the right-hand side.

8. Do exercise 2.8.9 in the text. Don't waste energy trying to think of a new example. Look at the clever answer in the back of the book and show that

   (a) $P(X = 1, Y = 1) = P(X = 1)P(Y = 1)$, but

   (b) X and Y are not independent.

   The purpose of this question was to set up the next one.

9. Do exercise 2.8.10 in the text. Hint: Consider all 4 possibilities. Make a 2×2 table. Fill in the information you know and then solve for the rest.

10. Do exercise 2.8.12 in the text. The answer is 1/3.

11. Do exercise 2.8.15 in the text.

12. Do exercise 2.8.18 in the text. An additional hint is to define $k_1 = \sum_y h(y)$ and $k_2 = \sum_x g(x)$. Then express the marginal probability mass functions in terms of k_1 and k_2.

13. Exercise 2.8.19 is just like 2.8.18, except for continuous random variables. You don't have to do it, but you know how; just integrate instead of adding. The question is, does the example of Problem 6 contradict this theorem? Answer Yes or No and briefly explain.

14. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_Y(y)$.

15. Let $X_1, \ldots, X_n$ be independent exponential random variables with parameter λ. Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_Y(y)$. Do not forget to indicate where the density is non-zero.

16. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$. Find the density $f_Y(y)$.

17. Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent. Using the convolution formula, find the probability mass function of $Z = X + Y$ and identify it by name.

18. Let X_1 and X_2 be independent exponential random variables with parameter $\lambda = 1$. Find the probability density function of $Y_1 = X_1/X_2$.

19. Let X_1 and X_2 be independent exponential random variables with parameter $\lambda = 1$. Find the probability density function of $Y_1 = \frac{X_1}{X_1+X_2}$. Be sure to specify where the density is non-zero.

20. Let X_1 and X_2 be independent standard normal random variables; that is, $\mu = 0$ and $\sigma^2 = 1$. Find the probability density function of $Y_1 = X_1/X_2$.

21. Show that the normal probability density function integrates to one. The formula for change to polar co-ordinates is on the formula sheet.
