Assignment 6

① For $x \geq 0$ $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$

$$= \int_0^x \lambda e^{-\lambda t}\,dt \qquad \begin{array}{l} u = -\lambda t \\ du = -\lambda\,dt \end{array} \qquad \begin{array}{c|c} t & u \\ \hline x & -\lambda x \\ 0 & 0 \end{array}$$

$$= -\int_0^{-\lambda x} e^u\,du = \int_{-\lambda x}^{0} e^u\,du = e^u \Big|_{-\lambda x}^{0}$$

$= 1 - e^{-\lambda x}$, so

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

② $X \sim N(\mu, \sigma^2)$ Using the fact (proved in earlier HW) that $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$,

$$F_X(x) = P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

3 2.5.7 $F_X(x) = x^2$ for $0 \le x \le 1$

(a) $P(X < \frac{1}{3}) = \frac{1}{9}$

(b) $P(\frac{1}{4} < X < \frac{1}{2}) = F_X(\frac{1}{2}) - F_X(\frac{1}{4}) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$

(c) $P(\frac{2}{5} < X < \frac{4}{5}) = F_X(\frac{4}{5}) - F_X(\frac{2}{5}) = \frac{16}{25} - \frac{4}{25} = \frac{12}{25}$

(d) $P(X < 0) = 0^2 = 0$

(e) $P(X < 1) = 1^2 = 1$

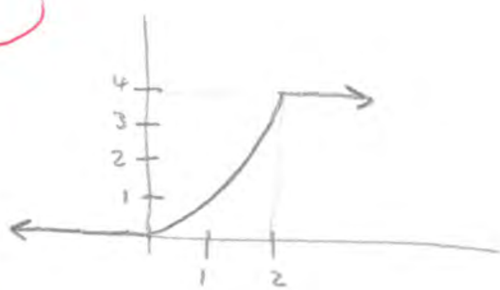
(f) $P(X < -1) = 0$

(g) $P(X < 3) = 1$

(h) $P(X = 3/7) = F(3/7) - F(3/7) = 0$

2.5.9

(a)



(b) No. Probabilities are between 0 & 1

2.5.21 Weibul $f_X(x) = \alpha x^{\alpha-1} e^{-x^\alpha}$ for $x > 0$. So for $x > 0$

$$F_X(x) = P(X \le x) = \int_0^x \alpha t^{\alpha-1} e^{-t^\alpha} dt \qquad u = t^\alpha \quad du = \alpha t^{\alpha-1} dt$$

t	u
x	x^α
0	0

$$= \int_0^{x^\alpha} e^{-u} du$$ This is the cdf of an Exponential (1) See problem 1

$$= 1 - e^{-x^\alpha}$$, so

$$F_X(x) = \begin{cases} 1 - e^{-x^\alpha} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

3 continued

2.5.23 Cauchy: $f_X(x) = \frac{1}{\pi(1+x^2)}$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,dt = \frac{1}{\pi}\tan^{-1}(t)\Big|_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left(\tan^{-1}(x) - \lim_{t\to-\infty}\tan^{-1}(t)\right)$$

$$= \frac{1}{\pi}\left(\tan^{-1}(x) - -\frac{\pi}{2}\right) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$$

(4) The text proves these properties using nested sets. It can also be done using the ~~axioms~~ basic properties of probability, as follows:

(a) Let $A_1 = \{s \in S : X(s) \leq 1\}$

$$A_2 = \{s \in S : 1 < X(s) \leq 2\}$$

$$\vdots$$

$$A_n = \{s \in S : n-1 < X(s) \leq n\}$$

$A_1 \ldots A_n$ are disjoint, and so by Property 4,

$$F_X(n) = P(X \leq n) = P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$$

Also, $\bigcup_{k=1}^{\infty} A_k = S$, disjoint, so again, by Prop 4, and 2,

$$1 = P(S) = P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) = \lim_{n\to\infty} P(X \leq n)$$

$$= \lim_{n\to\infty} F_X(n) = \lim_{x\to\infty} F_X(x) \quad \blacksquare$$

It might be easier to define $A_n = \{s \in S : X(s) \leq n\}$.
Note $A_1 \subseteq A_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} A_n = S$, so that

$$\lim_{x\to\infty} F_X(x) = \lim_{n\to\infty} F_X(n) = \lim_{n\to\infty} P(A_n) = P(S) = 1$$

Yes, that was easier

(4b) Show $\lim_{x \to -\infty} F_X(x) = 0$

Let $A_1 = \{s \in S : X(s) \le -1\}$

$A_2 = \{s \in S : X(s) \le -2\}$

$\vdots$

$A_n = \{s \in S : X(s) \le -n\}$

Have $A_1 \supseteq A_2 \supseteq \cdots$, and $\bigcap_{k=1}^{\infty} A_k = \emptyset$, so

$$\lim_{x \to -\infty} F_X(x) = \lim_{n \to \infty} F_X(-n) = \lim_{n \to \infty} P(A_n)$$

$$= P\left(\bigcap_{k=1}^{\infty} A_k\right) = P(\emptyset) = 0 \quad \boxtimes$$

Assignment 6

(5) (4)

2.6.1

$X \sim U(L,R)$, $Y = cX+d$ where $c>0$

$L \le X \le R \iff cL+d \le \underbrace{cX+d}_{Y} \le cR+d$, so for $cL+d \le y \le cR+d$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(cX+d \le y) = \frac{d}{dy} P(cX \le y-d)$$

$$= \frac{d}{dy} P\left(X \le \frac{y-d}{c}\right) = \frac{d}{dy} F_X\left(\frac{y-d}{c}\right) = f_X\left(\frac{y-d}{c}\right) \cdot \frac{1}{c}$$

$$= \frac{1}{c} \frac{1}{R-L} = \frac{1}{cR-cL} \text{. So}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{cR+d-(cL+d)} & \text{for } cL+d \le y \le cR+d \\ 0 & \text{otherwise} \end{cases}$$

That is, $Y \sim U(cL+d, cR+d)$

2.6.3

$X \sim N(\mu, \sigma^2)$ $Y = cX+d$, $c>0$

$$f_Y(y) = \dots = f_X\left(\frac{y-d}{c}\right) \cdot \frac{1}{c} \text{ as in 2.6.1}$$

$$= \frac{1}{c} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(\frac{y-d}{c} - \mu\frac{c}{c}\right)^2}$$

$$= \frac{1}{c\sigma\sqrt{2\pi}} e^{-\frac{1}{2c^2\sigma^2}\left(y-(c\mu+d)\right)^2}$$

That is, $Y \sim N(c\mu+d, c^2\sigma^2)$

2.6.4 $X \sim Exp(\lambda)$, $Y = cX$ where $c > 0$

Note $X \geq 0 \iff Y = cX \geq 0$, so for $y \geq 0$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(cX \leq y)$$

$$= \frac{d}{dy} P\left(X \leq \frac{y}{c}\right) = \frac{d}{dy} F_X\left(\frac{y}{c}\right) = f_X\left(\frac{y}{c}\right) \cdot \frac{1}{c}$$

$$= \frac{1}{c} \cdot \lambda e^{-\lambda \frac{y}{c}}$$, and

$$f_Y(y) = \begin{cases} \frac{\lambda}{c} e^{-\frac{\lambda}{c} y} & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

That is, $Y \sim Exp\left(\frac{\lambda}{c}\right)$

2.6.5 $X \sim Exp(\lambda)$, $Y = X^3$

Note $X \geq 0 \iff Y = X^3 \geq 0$, so for $y \geq 0$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(X^3 \leq y)$$

$$= \frac{d}{dy} P(X \leq y^{1/3}) = \frac{d}{dy} F_X(y^{1/3})$$

$$= f_X(y^{\frac{1}{3}}) \cdot \frac{1}{3} y^{-\frac{2}{3}} = \lambda e^{-\lambda y^{1/3}} \cdot \frac{1}{3} y^{-2/3}$$, and

$$f_Y(y) = \begin{cases} \frac{\lambda}{3} e^{-\lambda y^{1/3}} y^{-2/3} & \text{for } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

c

2.6.7 $X \sim U(0,3)$, $Y = X^2$

Note $0 \le X \le 3 \iff 0 \le X^2 \le 9$, i.e. $0 \le Y \le 9$, so
for $y \in [0,9]$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(X^2 \le y)$$

$$= \frac{d}{dy} P(X \le y^{1/2}) = \frac{d}{dy} F_X(y^{\frac{1}{2}}) = f_X(y^{\frac{1}{2}}) \cdot \frac{1}{2} y^{-\frac{1}{2}}$$

$$= \frac{1}{3} \cdot \frac{1}{2} y^{-1/2}, \text{ and } f_Y(y) = \begin{cases} \frac{1}{6\sqrt{y}} & \text{for } 0 \le y \le 9 \\ 0 & \text{otherwise} \end{cases}$$

2.6.8 $f_X(\mu + x) = f_X(\mu - x)$, $Y = 2\mu - X$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(2\mu - X \le y)$$

$$= \frac{d}{dy} P(-X \le y - 2\mu) = \frac{d}{dy} P(X \ge 2\mu - y)$$

$$= \frac{d}{dy}\left(1 - F_X(2\mu - y)\right) = -f_X(2\mu - y)(-1)$$

$$= f_X(\mu + \underbrace{\mu - y}_{\text{a new } x}) = f_X(\mu - (\mu - y))$$

$$= f_X(\mu - \mu + y) = f_X(y)$$

Pretty cute

(6) $P_{XY}(x, y) = c(x+y)$ for $x = 1, 2, 3$ & $y = 1, 2$

	$x=1$	$x=2$	$x=3$	
$y=2$	$3/21$	$4/21$	$5/21$	$\frac{12}{21}$
$y=1$	$2/21$	$3/21$	$4/21$	$\frac{9}{21}$
	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$	

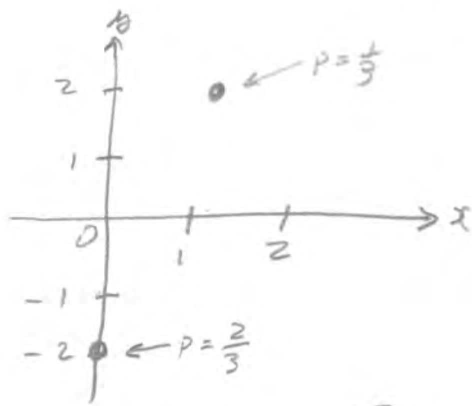
The numbers sum to 21

Note $F_{XY}(x, y)$ adds up all the probability down and to the left.

(7) (2.7.1) $X \sim \text{Bernoulli}(\frac{1}{3})$, $Y = 4X - 2$

If $X = 0$, $Y = -2$ If $X = 1$, $Y = +2$

(a)



(b)

$$P_{XY}(x, y) = \begin{cases} \frac{1}{3} & \text{for } x = 1, y = 2 \\ \frac{2}{3} & \text{for } x = 0, y = -2 \\ 0 & \text{otherwise} \end{cases}$$

(c) $F_{XY}(1, 1) = P(X \leq 1, Y \leq 1) = P(X = 0, Y = -2)$

Down and to the left

8 2.7.3

(C)

$y \backslash x$	-3	-2	2	3	17	
19					$\frac{1}{5}$	$\frac{1}{5}$
3			$\frac{1}{5}$			$\frac{1}{5}$
2				$\frac{1}{5}$		$\frac{1}{5}$
-2	$\frac{1}{5}$					$\frac{1}{5}$
-3		$\frac{1}{5}$				$\frac{1}{5}$
	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1

a) $P_X(x) = \begin{cases} \frac{1}{5} & \text{for } x = -3, -2, 2, 3, 17 \\ 0 & \text{otherwise} \end{cases}$

b) $P_Y(y) = \begin{cases} \frac{1}{5} & \text{for } y = -3, -2, -2, 3, 19 \\ 0 & \text{otherwise} \end{cases}$

c) $P(Y > X) = P(X=-3, Y=-2) + P(X=2, Y=3) + P(X=17, Y=\frac{1}{9}) = \frac{3}{5}$

d) $P(Y=X) = 0$

e) $P(XY) < 0 = 0$

(9) 2.7.5

Since $X \le x$ & $Y \le y$ implies $X \le x$,

$$\{s \in S : X(s) \le x \text{ \& } Y(s) \le y\} \subseteq \{s \in S : X(s) \le x\}$$

$$\Rightarrow P(X \le x, Y \le y) \le P(X \le x)$$

$$\underset{F_{XY}(x,y)}{\parallel} \qquad \underset{F_X(x)}{\parallel}$$

A similar argument shows $F_{XY}(x,y) \le F_Y(y)$. So, $F_{XY}(x,y)$ is smaller than $F_X(x)$ & smaller than $F_Y(y)$.

$$\Rightarrow F_{XY}(x,y) \le \min\left(F_X(x), F_Y(y)\right)$$

(10) 5 cards, X = # of spades, Y = # of Hearts

(a) Without replacement

$$P_{XY}(x,y) = P(X=x, Y=y) = \begin{cases} \dfrac{\binom{13}{x}\binom{13}{y}\binom{26}{5-x-y}}{\binom{52}{5}} & \text{for } x \ge 0, y \ge 0, x+y \le 5 \text{ (all integers)} \\ 0 & \text{otherwise} \end{cases}$$

(b) With replacement It's multinomial

$$P_{XY}(x,y) = P(X=x, Y=y) = \begin{cases} \binom{5}{x \;\; y \;\; 5-x-y} \left(\frac{1}{4}\right)^x \left(\frac{1}{4}\right)^y \left(\frac{1}{2}\right)^{5-x-y} & \text{same support above} \\ 0 & \text{otherwise} \end{cases}$$

11 F_{XY} on p. 82

(a) $f_{XY}(x,y) = \frac{\partial^2 F}{\partial x \partial y}$ Do it in pieces

If ($x<0$ or $y<0$) or ($x>1$ and $y>1$), derivative of a constant is zero

If $0 \le x \le 1$ and $0 \le y \le 1$ $\frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y}(xy^2)$

$= \frac{\partial}{\partial x} x 2y = 2y$

If $0 \le x \le 1$, $y \ge 1$ $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} x = \frac{\partial}{\partial x} 0 = 0$

If $x \ge 1$, $0 \le y \le 1$ $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} y^2 = \frac{\partial}{\partial x} 2y = 0$

so $f_{XY}(x,y) = \begin{cases} 2y & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

(b) $\int_0^1 \int_0^1 2y \, dx\, dy = \int_0^1 2y \underbrace{\int_0^1 dx}_{=1} dy = 2\int_0^1 y\, dy$

$= 2 \frac{y^2}{2} \Big|_0^1 = 1 - 0 = 1$

(9)

(11c) For $0 \le x \le 1$, $f_X(x) = \int_0^1 2y\,dy = 2\frac{y^2}{2}\Big|_0^1 = 1$

So $f_X(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ Uniform(0,1)

(d) for $0 \le y \le 1$, $f_Y(y) = \int_0^1 2y\,dx = 2y\int_0^1 1\,dx$
$= 2y$, so

$$f_Y(y) = \begin{cases} 2y & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(e) Find $F_X(x)$

(i) For $0 \le x \le 1$, $P(X \le x) = \int_0^x f_X(t)\,dt = \int_0^x 1\,dt$

$= x$, so $F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$

Or you could do each piece separately

(ii) For $x < 0$, $\lim_{y\to\infty} F_{XY}(x,y) = \lim_{y\to\infty} 0 = 0$

For $0 \le x \le 1$, $\lim_{y\to\infty} F_{XY}(x,y) = \lim_{y\to\infty} x = x$

For $x > 1$, $\lim_{y\to\infty} F_{XY}(x,y) = \lim_{y\to\infty} 1 = 1$, and

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

(11f) Find $F_Y(y)$

(i) By integrating. If $0 \le y \le 1$, $F_Y(y) = \int_0^y 2t\,dt$

$$= 2\frac{t^2}{2}\Big|_0^y = y^2, \text{ so}$$

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

ii) If $y < 0$, $F_Y(y) = \lim_{x \to \infty} F_{XY}(x,y) = \lim_{x\to\infty} 0 = 0$

If $0 \le y \le 1$, $F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y)$

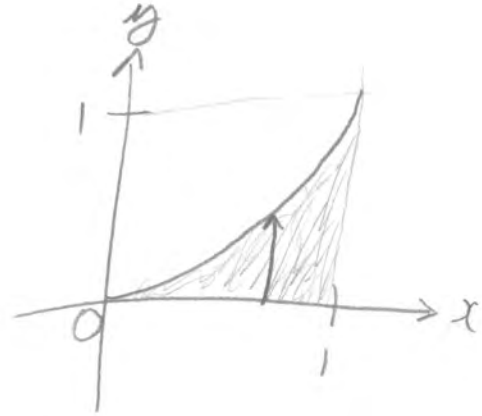
$$= \lim_{x\to\infty} y^2 = y^2$$

If $y > 1$, $F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y) = \lim_{x\to\infty} 1 = 1$, so

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

(11g) $P(Y < X^2)$

(j)



$$= \int_0^1 \int_0^{x^2} 2y \, dy \, dx$$

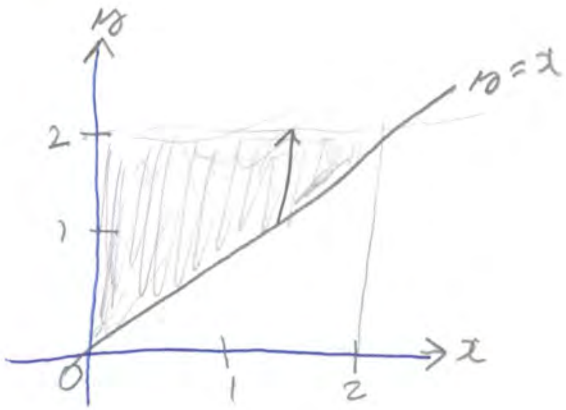
$$= 2\int_0^1 \int_0^{x^2} y \, dy \, dx$$

$$= \cancel{2}\int_0^1 \frac{y^2}{\cancel{2}} \Big|_0^{x^2} dx = \int_0^1 x^4 \, dx = \frac{x^5}{5} \Big|_0^1$$

$$= \frac{1}{5}$$

(2)

12 2.7.9 $f_{XY}(x,y) = \begin{cases} \frac{1}{4}(x^2+y) & \text{for } 0<x<y<2 \\ 0 & \text{otherwise} \end{cases}$



(a) For $0<x<2$, $f_X(x) = \int_x^2 \frac{1}{4}(x^2+y)\,dy$

$$= \frac{1}{4}\left(\int_x^2 x^2\,dy + \int_x^2 y\,dy\right) = \frac{1}{4}\left(x^2\int_x^2 dy + \frac{y^2}{2}\Big|_x^2\right)$$

$$= \frac{1}{4}\left(x^2(2-x) + \frac{1}{2}(4-x^2)\right)$$

$$= \frac{1}{4}\left(2x^2 - x^3 + 2 - \frac{1}{2}x^2\right) = \frac{1}{8}\left(4x^2 - 2x^3 + 4 - x^2\right)$$

$$= \frac{1}{8}\left(3x^2 - 2x^3 + 4\right), \text{ so}$$

$$f_X(x) = \begin{cases} \frac{1}{8}(3x^2-2x^3+4) & \text{for } 0<x<2 \\ 0 & \text{otherwise} \end{cases}$$

(2.7.9 b) For $0<y<2$,

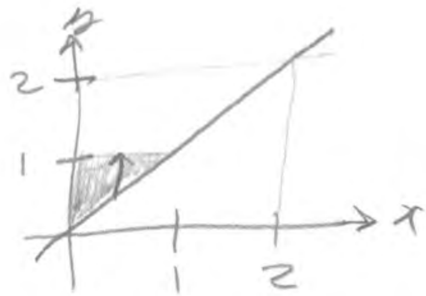
(2)

$$f_Y(y) = \int_0^y \frac{1}{4}(x^2+y)\,dx$$

$$= \frac{1}{4}\left(\int_0^y x^2\,dx + y\int_0^y 1\,dx\right) = \frac{1}{4}\left(\frac{x^3}{3}\Big|_0^y + yx\Big|_0^y\right)$$

$$= \frac{1}{4}\left(\frac{y^3}{3} + y^2\right) = \frac{1}{12}\left(y^3+3y^2\right), \text{ so}$$

$$f_Y(y) = \begin{cases} \frac{1}{12}\left(y^3+3y^2\right) & \text{for } 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$



(c) $P(Y<1)$

$$= \int_0^1\int_x^1 \frac{1}{4}(x^2+y)\,dy\,dx$$

$$= \frac{1}{4}\int_0^1\left[x^2\int_x^1 dy + \int_x^1 y\,dy\right]dx = \frac{1}{4}\int_0^1 x^2(1-x) + \frac{y^2}{2}\Big|_x^1\,dx$$

$$= \frac{1}{4}\int_0^1\left(x^2-x^3+\frac{1}{2}(1-x^2)\right)dx = \frac{1}{4}\int_0^1\left(x^2-x^3+\frac{1}{2}-\frac{x^2}{2}\right)dx$$

$$= \frac{1}{4}\int_0^1\left(\frac{x^2}{2}-x^3+\frac{1}{2}\right)dx = \frac{1}{8}\int_0^1\left(x^2-2x^3+1\right)dx$$

$$= \frac{1}{8}\left(\frac{x^3}{3}-\frac{2x^4}{4}+x\Big|_0^1\right) = \frac{1}{8}\left(\frac{1}{3}-\frac{1}{2}+1\right) = \frac{1}{8}\left(\frac{2}{6}-\frac{3}{6}+\frac{6}{6}\right)$$

$$= \frac{1}{48}(5) = 5/48$$

(13) 2.7.12 Show $\lim_{x\to-\infty} F_{XY}(x,y) = 0$

Because $F_{XY}(x,y)$ is a probability, it is non-negative

By Ex. 2.7.5

$$0 \le F_{XY}(x,y) \le F_X(x)$$

$$\Rightarrow 0 \le \lim_{x\to-\infty} F_{XY}(x,y) \le \lim_{x\to-\infty} F_X(x) = 0$$ squeeze

14 2.7.16 $f_{XY}(x,y) = c\,e^{-(x+y)}$ for $0 < x < y < \infty$

(a) $1 = c\int_0^{\infty}\int_x^{\infty} e^{-x}e^{-y}\,dy\,dx = c\int_0^{\infty} e^{-x}\underbrace{\int_x^{\infty} e^{-y}\,dy}\,dx$

$1-F(x)$ for a standard exponential

$= c\int_0^{\infty} e^{-x}e^{-x}\,dx = \frac{c}{2}\underbrace{\int_0^{\infty} 2e^{-2x}\,dx}_{=1} = 1$

$\Rightarrow c = 2$

(b) For $x \geq 0$, $f_X(x) = \int_x^{\infty} 2e^{-x}e^{-y}\,dy$

$= 2e^{-x}\int_x^{\infty} e^{-y}\,dy = 2e^{-x}e^{-x} = 2e^{-2x}$, so

$f_X(x) = \begin{cases} 2e^{-2x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ Exponential, $\lambda = 2$

For $y \geq 0$
$f_Y(y) = \int_0^{y} 2e^{-x}e^{-y}\,dx = 2e^{-y}\int_0^{y} e^{-x}\,dx$

$= 2e^{-y}(1-e^{-y})$, so

$f_Y(y) = \begin{cases} 2(e^{-y} - e^{-2y}) & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$